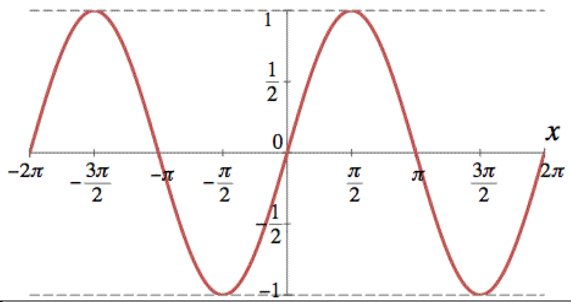
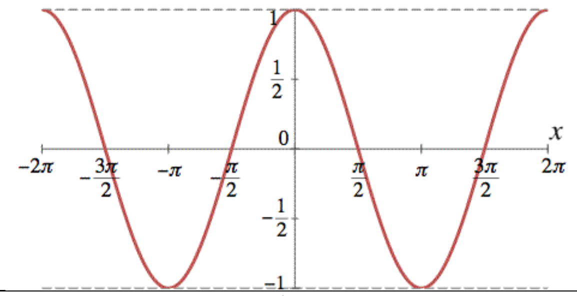
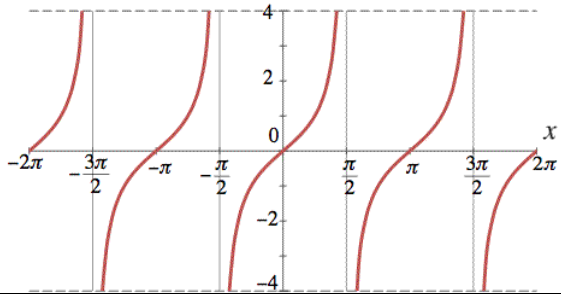
Calculus 1 & 2 Subject Notes

Trigonometric Functions

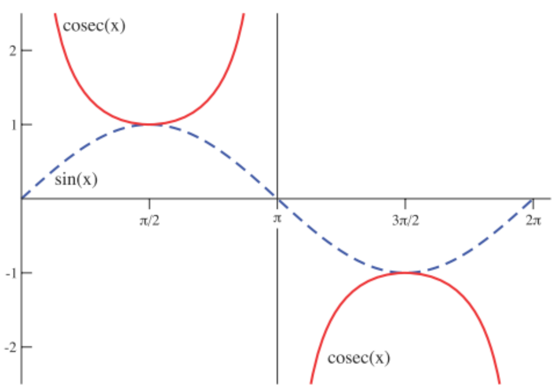
Trigonometric functions

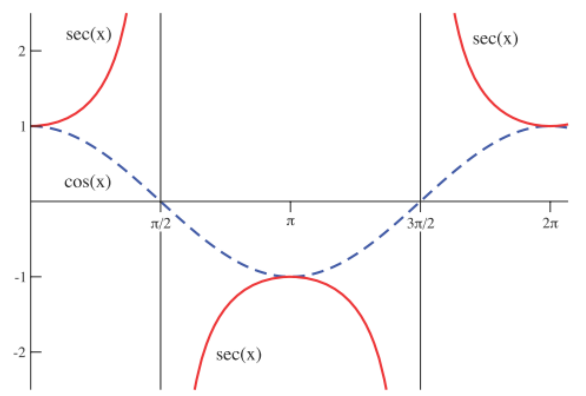
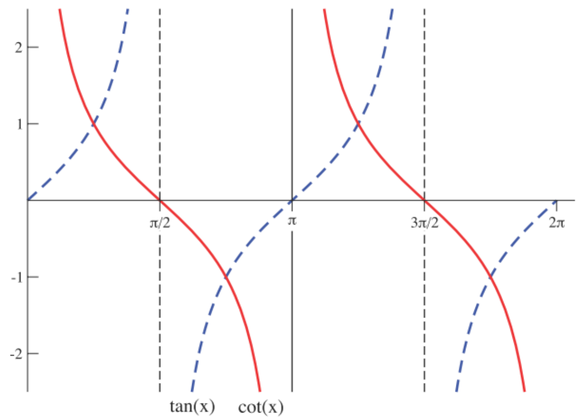
Readers will be familiar with the standard trigonometric functions.

Formula	Domain and Range	Graph
$\sin(x)$	$D: \mathbb{R}$ $R: \mathbb{R}$	
$\cos(x)$	$D: \mathbb{R}$ $R: \mathbb{R}$	
$\tan(x) = \dfrac{\sin(x)}{\cos(x)}$	$D: \mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$ $R: \mathbb{R}$	

Reciprocal trigonometric functions

The basic trigonometric functions all have reciprocals that also have distinct names.

Formula	Domain and Range	Graph
$\text{cosec}(x) = \dfrac{1}{\sin(x)}$	$D: \mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$ $R: \mathbb{R} \setminus (-1,1)$	

$\sec(x) = \dfrac{1}{\cos(x)}$	$D: \mathbb{R} \setminus \left\{\dfrac{\pi}{2} + n\pi : n \in \mathbb{Z}\right\}$ $R: \mathbb{R} \setminus (-1,1)$	
$\cot(x) = \dfrac{1}{\tan(x)}$ $= \dfrac{\cos(x)}{\sin(x)}$	$D: \mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$ $R: \mathbb{R}$	

Trigonometric identities

Pythagorean identity

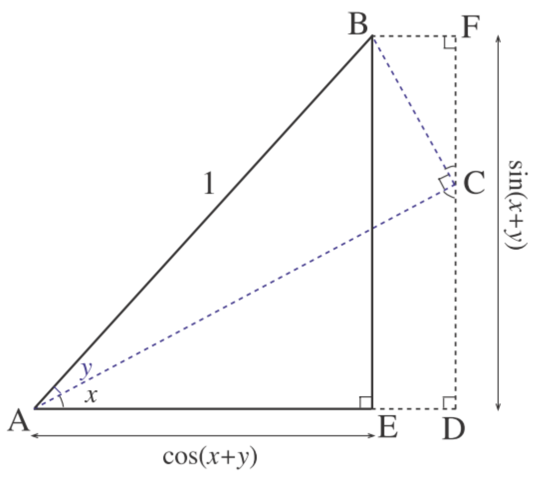
$$a^2 + b^2 = c^2 \rightarrow \sin^2(\theta) + \cos^2(\theta) = 1$$

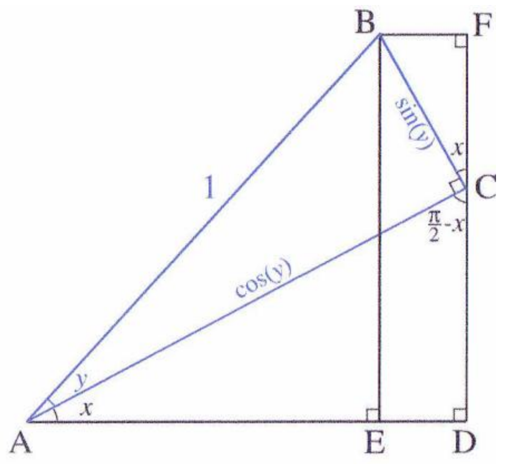
$$\frac{\sin^2(\theta)}{\cos^2(\theta)} + 1 = \frac{1}{\cos^2(\theta)} \rightarrow \tan^2(\theta) + 1 = \sec^2(\theta)$$

$$1 + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)} \rightarrow 1 + \cot^2(\theta) = \mathrm{cosec}^2(\theta)$$

Compound angle formulae

These identities can be derived by considering the following diagrams.





We can calculate the side lengths as follows:

$$\sin(x) = \frac{DC}{AC} \qquad \cos(x) = \frac{AD}{AC} \qquad \sin(x) = \frac{BF}{CB} \qquad \cos(x) = \frac{CF}{CB}$$
$$\sin(x) = \frac{DC}{\cos(y)} \qquad \cos(x) = \frac{AD}{\cos(y)} \qquad \sin(x) = \frac{BF}{\sin(y)} \qquad \cos(x) = \frac{CF}{\sin(y)}$$
$$\therefore DC = \sin(x)\cos(y) \qquad \therefore AD = \cos(x)\cos(y) \qquad \therefore BF = \sin(x)\sin(y) \qquad \therefore CF = \cos(x)\sin(y)$$

This allows us to derive the first two identities:

$$\begin{aligned} \sin(x+y) &= DF \\ &= DC + CF \\ \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \end{aligned}$$

$$\begin{aligned} \cos(x+y) &= AE \\ &= AD - DE \\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \end{aligned}$$

Using $\sin(-y) = -\sin(y)$ and $\cos(-y) = \cos(y)$ we find:

$$\begin{aligned} \sin(x-y) &= \sin\big(x + (-y)\big) \\ &= \sin(x)\cos(-y) + \cos(x)\sin(-y) \\ \sin(x-y) &= \sin(x)\cos(y) - \cos(x)\sin(y) \end{aligned}$$

$$\begin{aligned} \cos(x-y) &= \cos\big(x + (-y)\big) \\ &= \cos(x)\cos(-y) + \sin(x)\sin(-y) \\ \cos(x-y) &= \cos(x)\cos(y) + \sin(x)\sin(y) \end{aligned}$$

We can also find similar identities for tan:

$$\begin{aligned} \tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)} \\ &= \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(x)\cos(y) - \sin(x)\sin(y)} \\ &= \frac{\frac{\sin(x)\cos(y)}{\cos(x)\cos(y)} + \frac{\cos(x)\sin(y)}{\cos(x)\cos(y)}}{1 - \frac{\sin(x)\sin(y)}{\cos(x)\cos(y)}} \\ \tan(x+y) &= \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)} \end{aligned}$$

$$\begin{aligned} \tan(x-y) &= \frac{\sin(x-y)}{\cos(x-y)} \\ &= \frac{\sin(x)\cos(y) - \cos(x)\sin(y)}{\cos(x)\cos(y) + \sin(x)\sin(y)} \\ &= \frac{\frac{\sin(x)\cos(y)}{\cos(x)\cos(y)} - \frac{\cos(x)\sin(y)}{\cos(x)\cos(y)}}{1 + \frac{\sin(x)\sin(y)}{\cos(x)\cos(y)}} \\ \tan(x-y) &= \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)} \end{aligned}$$

The double angle formulae can be found using simple manipulations substituting $x = y$ above.

Table of compound angle formulae

$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x+y) = \dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$
$\tan(x-y) = \dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

Table of double angle formulae

$\sin(2x) = 2\sin(x)\cos(x)$
$\cos(2x) = \cos^2(x) - \sin^2(x)$
$\tan(2x) = \dfrac{2\tan(x)}{1-\tan^2(x)}$

Implied domain and range

When considering a composite function, the range of the inner function must intersect with the domain of the outer function. The implied domain of a composition function $f(g(x)) = (f \circ g)(x)$ is the set of all x in the domain of g such that $f(g(x))$ is defined. Similarly, the implied range is the set of all y for which $y = f(g(x))$ for some x.

For example, consider $h(x) = \sqrt{\log(x)}$. We have a range of $\mathbb{R}$ for the inner function and a domain of $[0, \infty)$ for the outer function. Thus the domain of the inner function must be restricted to $[1, \infty)$ to yield the required range of $[0, \infty)$. Hence the implied domain is $[1, \infty)$.

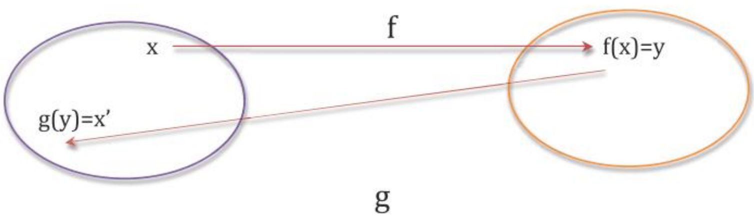
Inverse functions

If f is a function defined as:

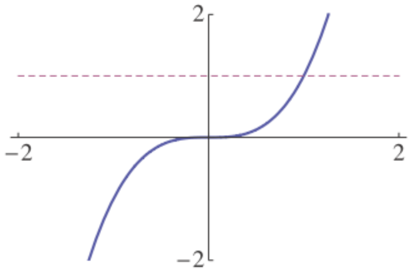
$$f: X \to Y, \qquad f(x) = y$$

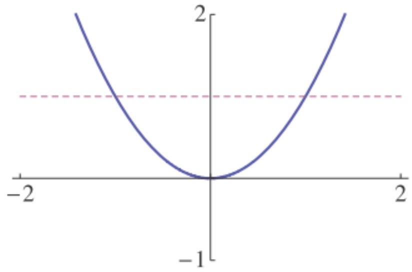
Then the inverse function g is defined as:

$$g: Y \to X, \qquad g(f(x)) = x$$



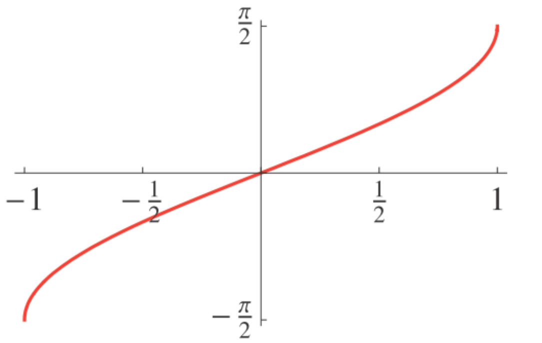
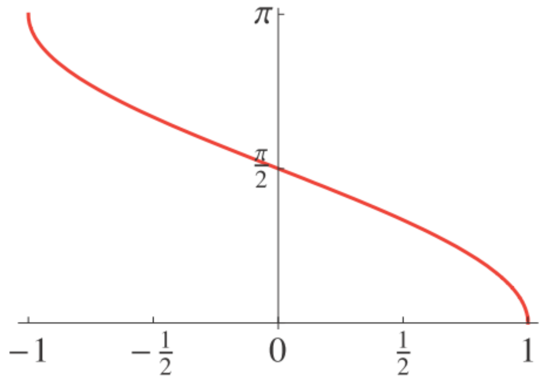
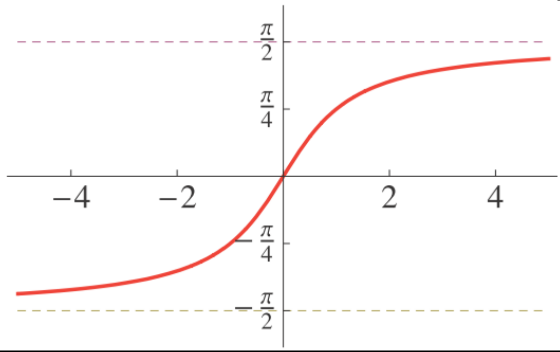
In order for an inverse function to exist, the initial function f must be one-to-one, meaning that each element of the codomain Y is only reached by a single element of the domain X. An easy way to determine if a function is one-to-one is if a horizontal line passes through a graph of the function only once.





Inverse trigonometric functions

Since all trigonometric functions are periodic they are obviously not one-to-one, and thus in order for an inverse function to exist the domain must be restricted to make them one-to-one. The standard restriction of the domain is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. With this restriction we can define the inverse trigonometric functions.

Formula	Domain and Range	Graph
$\theta = \arcsin(x)$	$D: [-1,1]$ $R: \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	
$\theta = \arccos(x)$	$D: [-1,1]$ $R: [0,\pi]$	
$\theta = \arctan(x)$	$D: \mathbb{R}$ $R: \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	

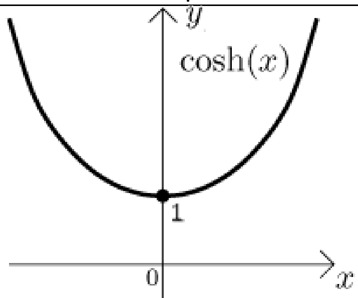
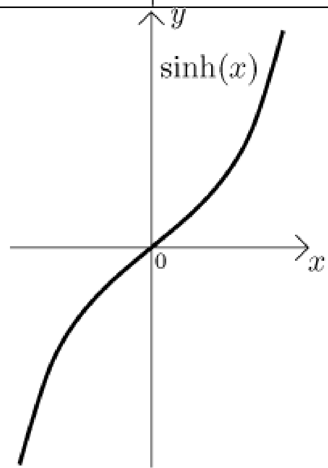
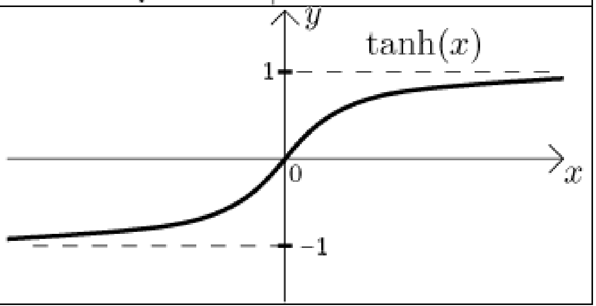
To evaluate an inverse trigonometric function we use the following technique:

$$\arccos\left(\sin\left(-\frac{\pi}{3}\right)\right) = \theta$$
$$\arccos\left(-\frac{\sqrt{3}}{2}\right) = \theta$$
$$-\frac{\sqrt{3}}{2} = \cos(\theta)$$
$$\therefore \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Hyperbolic trigonometric functions

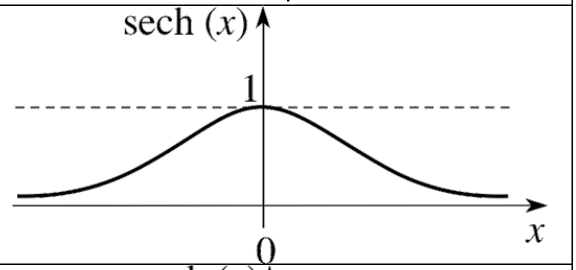
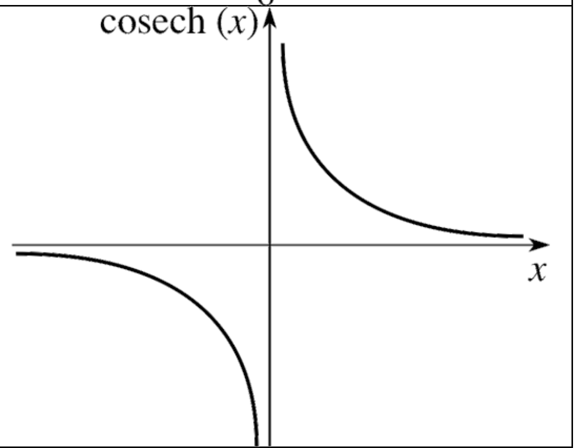
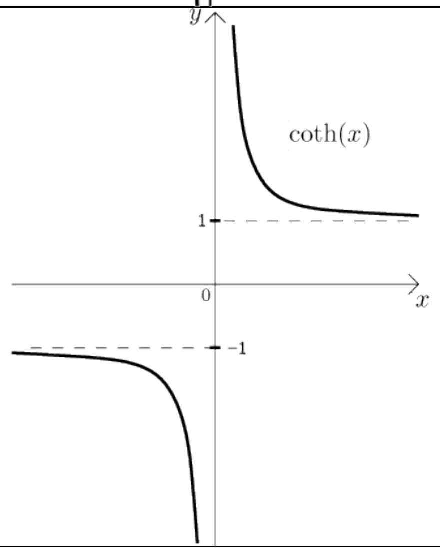
Hyperbolic functions are analogs of the ordinary trigonometric functions defined for a hyperbola rather than on a circle: just as the points (cos t, sin t) form a circle with a unit radius, the points (cosh t, sinh t) form the right half of the equilateral hyperbola. They obey the key identities:

$$\cosh^2(x) - \sinh^2(x) = 1$$
$$1 - \tanh^2(x) = \operatorname{sech}^2(x)$$

Formula	Domain and Range	Graph
$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$	$D: [-1,1]$ $R: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	
$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$	$D: [-1,1]$ $R: [0, \pi]$	
$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$	$D: \mathbb{R}$ $R: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	

Reciprocal hyperbolic trigonometric functions

The hyperbolic trigonometric functions also have reciprocals.

<table>
<tr><th>Formula</th><th>Domain and Range</th><th>Graph</th></tr>
<tr><td>$\operatorname{sech}(x) = \dfrac{1}{\cosh(x)}$
$\operatorname{sech}(x) = \dfrac{2}{e^x + e^{-x}}$</td><td>$D: \mathbb{R}$
$R: (0,1]$</td><td>
</td></tr>
<tr><td>$\operatorname{cosech}(x) = \dfrac{1}{2}(e^x - e^{-x})$
$\operatorname{cosech}(x) = \dfrac{2}{e^x - e^{-x}}$</td><td>$D: \mathbb{R} \setminus 0$
$R: \mathbb{R} \setminus 0$</td><td>
</td></tr>
<tr><td>$\coth(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$</td><td>$D: \mathbb{R}\setminus\{0\}$
$R: \mathbb{R} \setminus [-1,1]$</td><td>
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Hyperbolic trigonometric identities

Hyperbolic trigonometric identities mirror those of the regular trigonometric functions.

Table of hyperbolic compound angle formulae

$\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$
$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$
$\sinh(x - y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y)$
$\cosh(x - y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y)$

Table of hyperbolic double angle formulae

$\sinh(2x) = 2\sinh(x)\cosh(x)$
$\cosh(2x) = \cosh^2(x) + \sinh^2(x)$
$\cosh(2x) = 2\cosh^2(x) - 1$
$\cosh(2x) = 2\sinh^2(x) + 1$

Inverse hyperbolic trigonometric functions

The inverse hyperbolic trigonometric functions are unusual in that they can be written as logarithms.

$$\sinh(\operatorname{arcsinh}(x)) = x$$
$$\frac{1}{2}\left(e^{\operatorname{arcsinh}(x)} - e^{-\operatorname{arcsinh}(x)}\right) = x$$
$$e^{\operatorname{arcsinh}(x)} - e^{-\operatorname{arcsinh}(x)} = 2x$$
$$e^{2\operatorname{arcsinh}(x)} - 2xe^{\operatorname{arcsinh}(x)} - 1 = 0$$
$$\therefore e^{\operatorname{arcsinh}(x)} = \frac{2x}{2} \pm \frac{1}{2}\sqrt{4x^2 + 4}$$
$$e^{\operatorname{arcsinh}(x)} = x \pm \sqrt{x^2 + 1}$$
$$\boxed{\operatorname{arcsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)}$$

$$\tanh(\operatorname{arctanh}(x)) = x$$
$$\frac{e^{\operatorname{arctanh}(x)} - e^{-\operatorname{arctanh}(x)}}{e^{\operatorname{arctanh}(x)} + e^{-\operatorname{arctanh}(x)}} = x$$
$$e^{\operatorname{arctanh}(x)} - e^{-\operatorname{arctanh}(x)} = xe^{\operatorname{arctanh}(x)} + xe^{-\operatorname{arctanh}(x)}$$
$$e^{2\operatorname{arctanh}(x)} - 1 = xe^{2\operatorname{arctanh}(x)} + x$$
$$(1 - x)e^{2\operatorname{arctanh}(x)} - (1 + x) = 0$$
$$\therefore e^{\operatorname{arctanh}(x)} = \sqrt{\frac{1 + x}{1 - x}}$$
$$\boxed{\operatorname{arctanh}(x) = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right)}$$

Complex Numbers

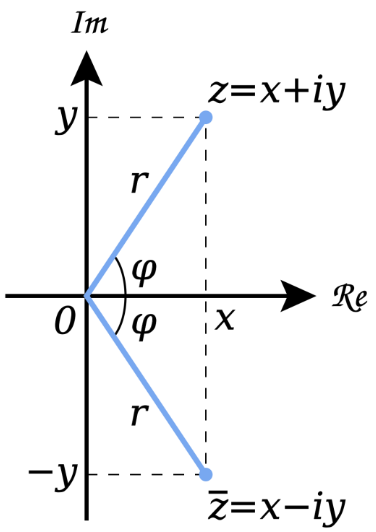
Introduction to complex numbers

A complex number z is a quantity consisting of a real number added to a multiple of the imaginary unit i:

$$z = x + iy$$

Where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$. The set of all complex numbers is denoted $\mathbb{C}$.

Complex numbers can be represented graphically using an Argand diagram (also called the complex plane), in which the real component x is plotted on the horizontal axis and the imaginary component y is plotted on the vertical axis.



Complex numbers z_1 and z_2 are equal if and only if both the real and imaginary components are equal:

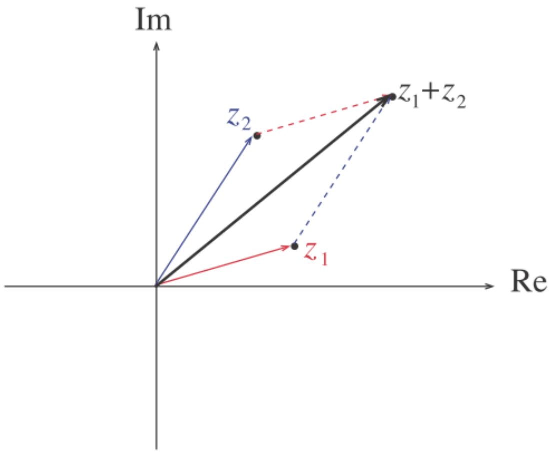
$$z_1 = z_2 \Leftrightarrow x_1 = x_2 \text{ and } y_1 = y_2$$

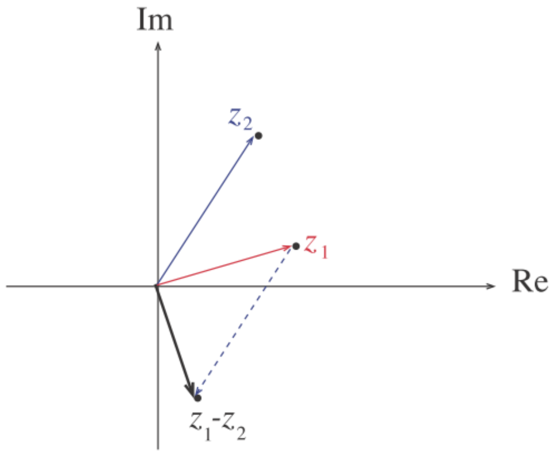
Operations on complex numbers

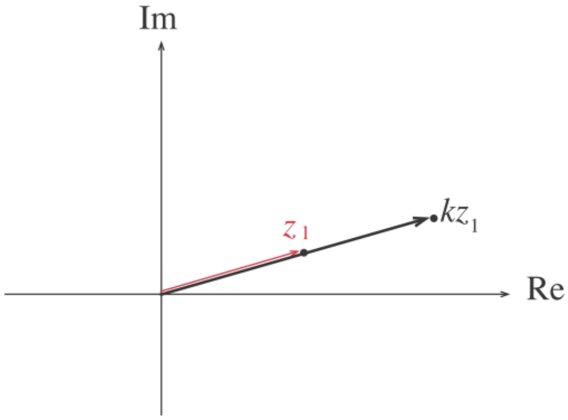
Complex numbers can be manipulated using standard operations as follows:

Addition	$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned}$
Subtraction	$\begin{aligned} z_1 - z_2 &= (x_1 + iy_1) - (x_2 + iy_2) \\ &= (x_1 - x_2) + i(y_1 - y_2) \end{aligned}$
Scalar multiplication	$\begin{aligned} kz_1 &= k(x + iy) \\ &= (kx) + i(ky) \end{aligned}$
Multiplication	$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + ix_2 y_1 + ix_1 y_2 + i^2 y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned}$

Addition, subtraction, and multiplication of complex numbers in the Argand plane is directly analogous to the same operations on vectors.







Complex conjugate

A new operation not defined for real numbers is called the complex conjugate. It is denoted by the bar notation ($\bar{z}$ 'z bar'), and is defined as keeping the real component and inverting the sign of the imaginary component. Hence:

$$\bar{z} = x_1 - iy_1$$

The complex conjugate is used in performing division of complex numbers. This is done by multiplying the denominator by its complex conjugate, so as to ensure that the denominator becomes real.

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} \\
&= \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2} \\
&= \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 - (iy_2)^2} \\
&= \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 - (iy_2)^2} \\
&= \frac{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)}{x_2^2 + y_2^2} \\
\frac{z_1}{z_2} &= \left(\frac{x_1x_2 - y_1y_2}{x_2^2 + y_2^2}\right) + \left(\frac{x_1y_2 + x_2y_1}{x_2^2 + y_2^2}\right) i
\end{aligned}$$

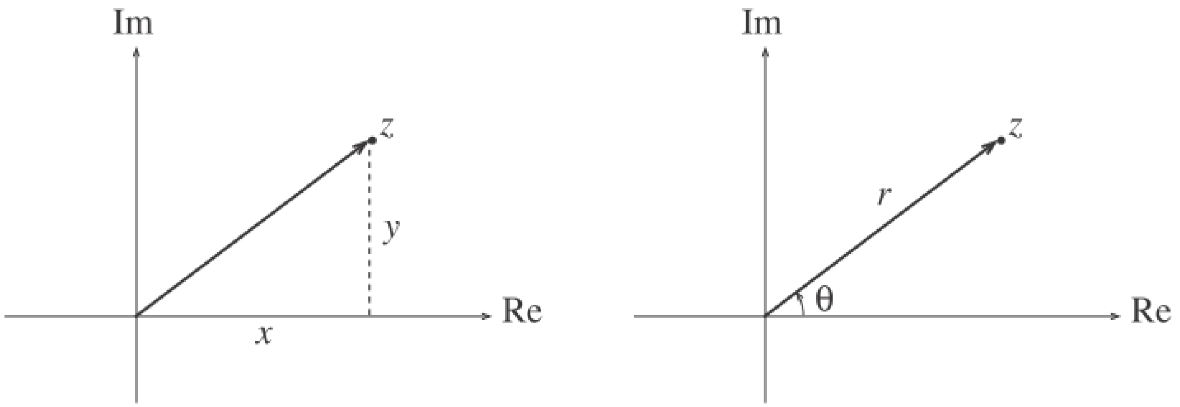
Complex polar form

Complex numbers can be represented in Cartesian form and polar form.

$$z = x + iy$$
$$\boxed{z = r\cos(\theta) + r\sin(\theta)}$$

This leads to a definition of the modulus of a complex number r, defined as the length of the complex number from the origin. It is calculated as:

$$|z| = r = \sqrt{x^2 + y^2}$$



Related to this is the argument of z, denoted $\arg(z)$ or simply θ, which is the angle in the counter-clockwise direction from the real axis that the complex number points in. The argument of z is not unique, since adding multiples of 2π does not change the position of z in the complex plane. However, there is only one value of the argument that satisfies $-\pi < \theta \leq \pi$, which is called the principal argument of z and is sometimes denoted Arg(z) with a capital A.

To convert a complex number from Cartesian to polar form, first calculate the modulus and then take the arctan of the ratio of x and y.

$$z = x_1 + iy_1$$
$$r = \sqrt{x_1^2 + y_1^2}$$
$$\theta = \arctan\left(\frac{y_1}{x_1}\right)$$

The polar form as the following properties:

$$|z_1 z_2| = |z_1||z_2|$$
$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$
$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$
$$\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$$

The complex exponential

The complex exponential is an equation that relates exponentials and trigonometric functions:

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$$

This equation seems very counterintuitive, however it can be shown fairly simply using the MacLaurin series for $\sin(x)$, $\cos(x)$, and e^z.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$
$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

If we substitute $z = ix$ into the exponential series we find:

$$\begin{aligned}
e^{ix} &= \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\
&= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots \\
&= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) \\
e^{ix} &= \cos(x) + i\sin(x)
\end{aligned}$$

The complex exponential has the following properties:

Zero angle	$e^{i0} = 1$
Multiplication	$e^{i(\theta_1)}e^{i(\theta_2)} = e^{i(\theta_1+\theta_2)}$
Division	$\frac{e^{i(\theta_1)}}{e^{i(\theta_2)}} = e^{i(\theta_1-\theta_2)}$

The complex exponential can be augmented with the modulus to define any complex number:

$$z = x + iy = r(\cos(x) + i\sin(y)) = re^{i\theta}$$

The complex exponential also gives rise to new definitions for $\sin\theta$ and $\cos\theta$.

$$\begin{aligned}
e^{i\theta} + e^{-i\theta} &= \cos(\theta) + i\sin(\theta) + \cos(\theta) + i\sin(-\theta) \\
&= 2\cos(\theta) + i(\sin(\theta) - \sin(\theta)) \\
&= 2\cos(\theta) \\
\cos(\theta) &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)
\end{aligned}$$

$$\begin{aligned}
e^{i\theta} - e^{-i\theta} &= \cos(\theta) + i\sin(\theta) - \cos(\theta) - i\sin(-\theta) \\
&= 2i\sin(\theta) \\
\sin(\theta) &= \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)
\end{aligned}$$

De Moivre's Theorem

De Moivre's theorem states that for any integer n:

$$\boxed{z^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i\sin(n\theta))}$$

This formula is very useful because it allows one to compute powers of complex numbers without expanding large numbers of brackets.

Example: find $\left(\frac{2}{1+i}\right)^{14}$ in exponential and Cartesian form.

$$\begin{aligned}
\left(\frac{2}{1+i}\right)^{14} &= 2^{14}(1+i)^{-14} \\
&= 2^{14}\left(\sqrt{2}\arctan\left(\frac{1}{1}\right)\right)^{-14} \\
&= 2^{14}\left(2^{\frac{1}{2}}e^{i\left(\frac{\pi}{4}\right)}\right)^{-14} \\
&= 2^{14}\left(2^{-7}e^{-14i\left(\frac{\pi}{4}\right)}\right) \\
&= 2^{7}e^{-i\left(\frac{7\pi}{2}\right)} \\
&= 2^{7}e^{i\left(\frac{\pi}{2}-\frac{8\pi}{2}\right)} \\
&= 2^{7}e^{i\left(\frac{\pi}{2}\right)} \\
&= 128\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) \\
&= 128\big(0 + i(1)\big) \\
\left(\frac{2}{1+i}\right)^{14} &= 128i
\end{aligned}$$

This method can also be applied in reverse to express trigonometric functions as sums of powers.

Example: express $\sin(3\theta)$ in terms of powers of $\sin(\theta)$.

$$\begin{aligned}
\sin(3\theta) &= \mathrm{Im}\left(e^{3i\theta}\right) \\
&= \mathrm{Im}\left[\left(e^{i\theta}\right)^3\right] \\
&= \mathrm{Im}[(\cos\theta + i\sin\theta)^3] \\
&= \mathrm{Im}[\cos^3(\theta) + 3\cos^2(\theta)\, i\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta)] \\
&= 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta) \\
&= 3(1-\sin^2(\theta))\sin(\theta) - \sin^3(\theta) \\
\sin(3\theta) &= 3\sin(\theta) - 4\sin^3(\theta)
\end{aligned}$$

Roots of complex numbers

Finding the roots of a complex number z means solving the equation (for $w \in \mathbb{C}$) :

$$z^n = w$$

Writing both numbers in exponential polar form this becomes:

$$\begin{aligned}
\left(re^{i\theta}\right)^n &= se^{i\phi} \\
r^n e^{in\theta} &= se^{i\phi}
\end{aligned}$$

We thus find the solution by equating the modulus:

$$r^n = s$$
$$\boxed{r = s^{\frac{1}{n}}}$$

And also by equating the argument:

$$n\theta = \phi + 2k\pi$$
$$\boxed{\theta = \frac{1}{n}(\phi + 2k\pi)}$$

Note that in order to find all n roots, we need $k = (0,1,2,\dots,n-1)$.

Example: find the 4th roots of $1 - \sqrt{3}i$.

$$\begin{aligned}
z^4 &= 1 - \sqrt{3}i \\
r^4 e^{i4\theta} &= 2e^{i\left(-\frac{\pi}{3}\right)} \\
\therefore r &= 2^{\frac{1}{4}} \\
\therefore 4\theta &= -\frac{\pi}{3} + 2\pi \times (0,1,2,3) \\
\theta &= -\frac{\pi}{12} + \frac{6\pi}{12} \times (0,1,2,3) \\
\theta &= -\frac{\pi}{12}, \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{-7\pi}{12} \\
\therefore z &= 2^{\frac{1}{4}}\left(e^{-\left(\frac{\pi}{12}\right)i}, e^{-\left(\frac{7\pi}{12}\right)i}, e^{\left(\frac{5\pi}{12}\right)i}, e^{\left(\frac{11\pi}{12}\right)i}\right) \\
z &= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)
\end{aligned}$$

The same method can also be used to find the roots of polynomials. The Fundamental Theorem of Algebra states that every polynomial of degree n can be factorised into n linear factors, using complex numbers.

$$P(z) = a(z - c_1)(z - c_2)\dots(z - c_n), \qquad \forall P(z)$$

Example: solve $z^4 - 2z^2 + 4 = 0$

$$\begin{aligned}
(z^2)^2 - 2(z^2) + 4 &= 0 \\
z^2 &= \frac{2 \pm \sqrt{4 - 16}}{2} \\
&= 1 \pm \frac{1}{2}\sqrt{-12} \\
&= 1 \pm \frac{1}{2}(2)\sqrt{-3} \\
&= 1 \pm \sqrt{-1 \times 3} \\
z^2 &= 1 \pm \sqrt{3}i \\
z^2 &= 2e^{\pm i\left(\frac{\pi}{3}\right)} \\
z &= \sqrt{2}e^{\pm\left[i\left(\frac{\pi}{6}\right) + \pi(0,1)\right]} \\
z &= \sqrt{2}e^{\frac{i\pi}{6}}, \sqrt{2}e^{\frac{i7\pi}{6}}, \sqrt{2}e^{-\frac{i\pi}{6}}, \sqrt{2}e^{-\frac{i7\pi}{6}}
\end{aligned}$$

Limits of Functions

Defining limits

A function $f(x)$ is said to have a limit $L \in \mathbb{R}$ as x approaches a if the value of the function approaches arbitrarily close to L whenever x is close enough, but not actually equal to, a. We denote this as follows.

$$\lim_{x \to a} f(x) = L$$

Note that a limit must have the same value approaching from either above or below (in the x axis), otherwise it does not exist. As such, even if the function is defined at a, the limit may not exist. Conversely, limits can exist even if the function is not defined at a.

Limit laws

Limits obey the following properties, so long as the limits of both functions f and g exist.

$$\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$
$$\lim_{x \to a}[cf(x)] = c \lim_{x \to a} f(x)$$
$$\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$
$$\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$
$$\lim_{x \to a}[c] = c$$
$$\lim_{x \to a}[x] = a$$
$$\lim_{x \to a}[x^n] = \left(\lim_{x \to a} x\right)^n$$

Techniques for evaluating limits

Factorisation

Some limits can be solved by factorisation, and cancellation of terms in the numerator and denominator.

Example: solve the following limit by factorisation.

$$\lim_{x \to 2}\left[\frac{x^2 - 4}{x - 2}\right] = \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)}$$
$$= \lim_{x \to 2}(x+2)$$
$$\lim_{x \to 2}\left[\frac{x^2 - 4}{x - 2}\right] = 4$$

Rationalisation

Some limits can be solved by rationalisation of the denominator, followed by cancellation of terms in the numerator and denominator.

Example: solve the following limit by rationalisation.

$$\lim_{x\to 1}\left[\frac{x-1}{\sqrt{x}-1}\right] = \lim_{x\to 1}\left[\frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)}\right]$$
$$= \lim_{x\to 1}\left[\frac{(x-1)(\sqrt{x}+1)}{(x-1)}\right]$$
$$= \lim_{x\to 1}(\sqrt{x}+1)$$
$$\lim_{x\to 1}\left[\frac{x-1}{\sqrt{x}-1}\right] = 2$$

Division of polynomials

Limits involving division of polynomials can sometimes by evaluated by dividing all terms by a term of order of the polynomial.

Example: solve the following limit using division.

$$\lim_{x\to\infty}\left[\frac{3x^2-2x+3}{x^2+4x+4}\right] = \lim_{x\to\infty}\left[\frac{3-\frac{2}{x}+\frac{3}{x^2}}{1+\frac{4}{x}+\frac{4}{x^2}}\right]$$
$$= \frac{\lim_{x\to\infty}\left[3-\frac{2}{x}+\frac{3}{x^2}\right]}{\lim_{x\to\infty}\left[1+\frac{4}{x}+\frac{4}{x^2}\right]}$$
$$= \frac{\lim_{x\to\infty}[3]-\lim_{x\to\infty}\left[\frac{2}{x}\right]+\lim_{x\to\infty}\left[\frac{3}{x^2}\right]}{\lim_{x\to\infty}[1]+\lim_{x\to\infty}\left[\frac{4}{x}\right]+\lim_{x\to\infty}\left[\frac{4}{x^2}\right]}$$
$$\lim_{x\to\infty}\left[\frac{3x^2-2x+3}{x^2+4x+4}\right] = 3$$

Sandwich theorem

This technique is useful for proving limits involving trigonometric functions. The theorem states that if $g(x) \le f(x) \le h(x)$ when x is near a (but not equal to a), then:

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L \implies \lim_{x\to a} f(x) = L$$

Example: solve the following limit using the sandwich theorem.

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$
$$-x^2 \le x^2\sin\left(\frac{1}{x}\right) \le x^2$$
$$-\lim_{x\to 0}[x^2] \le \lim_{x\to 0}\left[x^2\sin\left(\frac{1}{x}\right)\right] \le \lim_{x\to 0}[x^2]$$
$$0 \le \lim_{x\to 0}\left[x^2\sin\left(\frac{1}{x}\right)\right] \le 0$$
$$\therefore \lim_{x\to 0}\left[x^2\sin\left(\frac{1}{x}\right)\right] = 0$$

L'Hopital's rule

This technique is used when a limit involves the ratio of two limits which both tend to 0 or ∞. In such cases the rule states that:

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a}[f(x)]}{\lim_{x\to a}[g(x)]} = \frac{0}{0} \text{ or } \frac{\infty}{\infty} \quad \Longrightarrow \quad \lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \lim_{x\to a}\left[\frac{f'(x)}{g'(x)}\right]$$

Note that this only applies of the limits of the derivatives exist. This can also be applied multiple times, involving the second derivative, third derivative, etc.

Example: solve the following limit using L'Hopital's rule.

$$\begin{aligned}
\lim_{x\to\infty}\left[x^{-\frac{1}{3}}\ln(x)\right] &= \lim_{x\to\infty}\left[\frac{\ln(x)}{x^{\frac{1}{3}}}\right] \\
&= \lim_{x\to\infty}\left[\frac{\frac{d}{dx}(\ln(x))}{\frac{d}{dx}\left(x^{\frac{1}{3}}\right)}\right] \\
&= \lim_{x\to\infty}\left[\frac{\frac{1}{x}}{\frac{1}{3}\frac{1}{x^{\frac{2}{3}}}}\right] \\
&= \frac{\lim_{x\to\infty}\left[\frac{1}{x}\right]}{\frac{1}{3}\lim_{x\to\infty}\left[\frac{1}{x^{2/3}}\right]} \\
\lim_{x\to\infty}\left[x^{-\frac{1}{3}}\ln(x)\right] &= 0
\end{aligned}$$

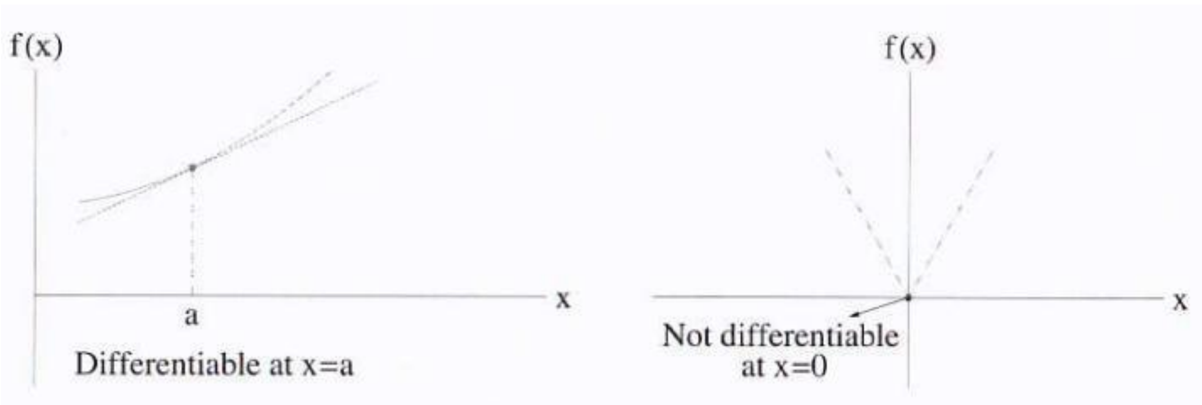
Differentiability

A function is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

The derivative of a function $f(x)$ at the point $x = a$ is defined by:

$$f'(x) = \lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right]$$

The function $f(x)$ is differentiable at $x = a$ if this limit exists. A function does not need to be continuous at a point in order to be differentiable at that point.



Differential Calculus

Implicit differentiation

Sometimes we want to find the derivative of one variable with respect to another even without an explicit functional form for the relationship. To do this we assume that one variable y depends implicitly on another variable x, then differentiate both sides of our equation with respect to x, and finally rearrange to solve for dy/dx.

Example: given $x^2 - xy + y^4 = 5$, find dy/dx.

$$
\begin{aligned}
x^2 - xy + y^4 &= 5 \\
\frac{d(x^2)}{dx} - \frac{d(xy)}{dx} + \frac{d(y^4)}{dx} &= 0 \\
2x - \left(y + x\frac{dy}{dx}\right) + 4y^3\left(\frac{dy}{dx}\right) &= 0 \\
2x - y - x\frac{dy}{dx} + 4y^3\left(\frac{dy}{dx}\right) &= 0 \\
\left(\frac{dy}{dx}\right)(4y^3 - x) &= y - 2x \\
\frac{dy}{dx} &= \frac{y - 2x}{4y^3 - x}
\end{aligned}
$$

Derivatives of inverse trigonometric functions

Differentiation of inverse trigonometric functions requires use of implicit differentiation. We begin with arcsin.

$$
\begin{aligned}
y &= \arcsin(x) \\
\sin(y) &= x \\
\frac{d(\sin(y))}{dx} &= 1 \\
\cos(y)\left(\frac{dy}{dx}\right) &= 1 \\
\frac{dy}{dx} &= \frac{1}{\cos(y)} \\
\frac{d\arcsin(x)}{dx} &= \frac{1}{\sqrt{1 - \sin^2(y)}} \\
\boxed{\frac{d\arcsin(x)}{dx} = \frac{1}{\sqrt{1 - x^2}}}
\end{aligned}
$$

We find the derivative of arccosine using a similar method.

$$
\begin{aligned}
y &= \arccos(x) \\
\cos(y) &= x \\
\frac{d(\cos(y))}{dx} &= 1 \\
-\sin(y)\left(\frac{dy}{dx}\right) &= 1 \\
\frac{dy}{dx} &= \frac{-1}{\sin(y)}
\end{aligned}
$$

$$\frac{d\,\arccos(x)}{dx} = \frac{-1}{\sqrt{1-\cos^2(y)}}$$

$$\boxed{\frac{\text{darccos}}{dx} = \frac{-1}{\sqrt{1-x^2}}}$$

Finally we find the derivative of arctan.

$$\begin{aligned}
y &= \arctan(x) \\
\tan(y) &= x \\
\frac{d(\tan(y))}{dx} &= 1 \\
\sec^2(y)\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\sec^2(y)} \\
\frac{d\,\arctan(x)}{dx} &= \frac{1}{\tan^2(y)+1}
\end{aligned}$$

$$\boxed{\frac{d\,\arctan(x)}{dx} = \frac{1}{1+x^2}}$$

Function extrema

Whether a function is increasing, constant, or decreasing over some domain is directly related to the sign of its derivative over that domain.

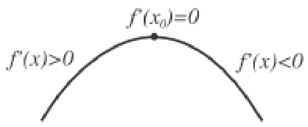
Theorem: If f is continuous over the interval $[a, b]$ and differentiable over (a, b), then

1. If $f'(x) > 0$ for all $x \in (a, b)$ then f is increasing over $[a, b]$.
2. If $f'(x) < 0$ for all $x \in (a, b)$ then f is decreasing over $[a, b]$.
3. If $f'(x) = 0$ for all $x \in (a, b)$ then f is constant over $[a, b]$.

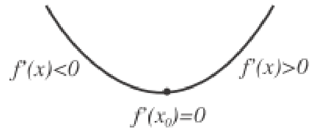
These results give rise to the relation between local maxima/local minima and the derivatives of a function. In particular, local maxima and minima are often stationary points, where the tangent to the function is horizontal. Specifically a stationary point occurs when $f'(x) = 0$.

The points where a function attains its overall largest and smallest values are called its **global extrema**.

A stationary point can be either a local maximum:



a local minimum:



or neither:

In the third case, the stationary point is called a *point of inflection*.

A related concept is that of concavity, which pertains to the second derivative of the function.

Theorem: Suppose f and f' are differentiable on an interval.

1. If $f''(x) > 0$ for every x in the interval, then f is concave up on the interval.

2. If $f''(x) < 0$ for every x in the interval, then f is concave down on the interval.

A point where f changes concavity has a special name:

Definition: A **point of inflection** is a point where f changes between being concave up and concave down.

Differentiation via the complex exponential

The complex exponential can be used to simplify differentiation of very high orders, if we can write the original function as a complex exponential with a linear function g of the argument θ.

$$\frac{d^n}{dt^n}\big(f(z)\big) = \frac{d^n}{dt^n}\big(re^{ig(\theta)}\big)$$

$$= r[ig(\theta)]^n e^{ig(\theta)}$$

$$\frac{d^n}{dt^n}\big(f(z)\big) = ri^n[g(\theta)]^n e^{ig(\theta)}$$

Example: Solve the following derivative using the complex exponential.

$$
\begin{aligned}
\frac{d^{56}}{dt^{56}}(e^{-t}\sin t) &= \frac{d^{56}}{dt^{56}}\mathrm{Im}\left[e^{-t}e^{it}\right] \\
&= \frac{d^{56}}{dt^{56}}\mathrm{Im}\left[e^{t(i-1)}\right] \\
&= \mathrm{Im}\left[\frac{d^{56}}{dt^{56}}e^{t(i-1)}\right] \\
&= \mathrm{Im}\left[(i-1)^{56}e^{t(i-1)}\right] \\
&= \mathrm{Im}\left[(-1+i)^{56}e^{t(i-1)}\right] \\
&= \mathrm{Im}\left[\left(\sqrt{2}e^{\left(\frac{3\pi}{4}\right)i}\right)^{56}e^{t(i-1)}\right] \\
&= \mathrm{Im}\left[\left(2^{28}e^{42\pi i}\right)e^{t(i-1)}\right] \\
\frac{d^{56}}{dt^{56}}(e^{-t}\sin t) &= 2^{28}(e^{-t}\sin t)
\end{aligned}
$$

Differentiation of hyperbolic trigonometric functions

Differentiation of hyperbolic trigonometric functions is similar to that of regular trigonometric functions.

We find the derivative of cosh and sinh in the same way.

$$
\begin{aligned}
\frac{d}{dx}(\cosh(x)) &= \frac{d}{dx}\left(\frac{1}{2}(e^{x}+e^{-x})\right) \\
&= \frac{1}{2}(e^{x}-e^{-x})
\end{aligned}
$$

$$\boxed{\frac{d}{dx}(\cosh(x)) = \sinh(x)}$$

$$
\begin{aligned}
\frac{d}{dx}(\sinh(x)) &= \frac{d}{dx}\left(\frac{1}{2}(e^{x}-e^{-x})\right) \\
&= \frac{1}{2}(e^{x}+e^{-x})
\end{aligned}
$$

$$\boxed{\frac{d}{dx}(\sinh(x)) = \cosh(x)}$$

The derivative of tanh is found as follows.

$$
\begin{aligned}
\frac{d}{dx}(\tanh(x)) &= \frac{d}{dx}\left(\frac{\sinh(x)}{\cosh(x)}\right) \\
&= \frac{\frac{d\sinh(x)}{dx}\cosh(x) - \frac{d\cosh(x)}{dx}\sinh(x)}{\cosh^2(x)} \\
&= \frac{\cosh^2(x)-\sinh^2(x)}{\cosh^2(x)} \\
&= \frac{1}{\cosh^2(x)}
\end{aligned}
$$

$$\boxed{\frac{d}{dx}(\tanh(x)) = \mathrm{sech}^2(x)}$$

Derivatives of the inverse hyperbolic functions can be found by implicit differentiation.

$$y = \operatorname{arcsinh}(x)$$
$$\sinh(y) = x$$
$$\frac{d(\sinh(y))}{dx} = 1$$
$$\cosh(y)\left(\frac{dy}{dx}\right) = 1$$
$$\frac{dy}{dx} = \frac{1}{\cosh(y)}$$
$$\frac{d\operatorname{arcsinh}(x)}{dx} = \frac{1}{\sqrt{1+\sinh^2(y)}}$$
$$\boxed{\frac{d\operatorname{arcsinh}(x)}{dx} = \frac{1}{\sqrt{x^2+1}}}$$

We find the derivative of arcosh using a similar method.

$$y = \operatorname{arccosh}(x)$$
$$\cosh(y) = x$$
$$\frac{d(\cosh(y))}{dx} = 1$$
$$\sinh(y)\left(\frac{dy}{dx}\right) = 1$$
$$\frac{dy}{dx} = \frac{1}{\sinh(y)}$$
$$\frac{d\arccos(x)}{dx} = \frac{1}{\sqrt{\cosh^2(y)-1}}$$
$$\boxed{\frac{d\mathrm{arccos}}{dx} = \frac{1}{\sqrt{x^2-1}}}$$

Finally we find the derivative of arctanh.

$$y = \operatorname{arctanh}(x)$$
$$\tanh(y) = x$$
$$\frac{d(\tanh(y))}{dx} = 1$$
$$\operatorname{sech}^2(y)\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2(y)}$$
$$\frac{d\operatorname{arctanh}(x)}{dx} = \frac{1}{\tanh^2(y)+1}$$
$$\boxed{\frac{d\operatorname{arctanh}(x)}{dx} = \frac{1}{1-x^2}}$$

Integral Calculus

Fundamental theorem of calculus

The Fundamental Theorem of Calculus describes the relationship between differentiation and integration. Specifically, it states the integral of the derivative of a function is equal to the original function, meaning that integration is equivalent to anti-differentiation.

$$f(x) = \int_a^x \frac{df}{dt} dt$$

Integration is linear:

$$\int af(x)\,dx = a\int f(x)\,dx$$
$$\int \big(f(x) + g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Integration is in general much more complicated than differentiation, in large part because there are no direct analogues to the product, quotient, and chain rules used in differentiation. Instead a wide range of techniques are used to solve more difficult integrals.

Substitutions

Integration by substitution requires knowledge of differentiation to make an informed guess about a substitution to make which will transform the integral into a form that is easier to solve. Specifically, if we can write the integrand in the form of the product of a composite function and the derivative of the inner part of that function, then the integral can be simplified greatly.

$$\int g\big(u(x)\big)\frac{du}{dx}\,dx = \int g(u)\,du$$

Example: solve the following integral:

$$\int \cos(3x)\sqrt{\sin(3x) + 4}\,dx$$

Here the outer function $g(u)$ will be $\sqrt{u}$, while the inner function will be $\sin(3x) + 4$. Hence:

$$u(x) = \sin(3x) + 4$$
$$g(u) = \sqrt{u}$$
$$\frac{du}{dx} = 3\cos(3x)$$

This yields:

$$\int \sqrt{u} \times \frac{1}{3}\frac{du}{dx}\,dx = \frac{1}{3}\int \sqrt{u}\,du$$
$$= \frac{1}{3}\frac{2}{3}u^{\frac{3}{2}}$$
$$\int \cos(3x)\sqrt{\sin(3x) + 4}\,dx = \frac{2}{9}(\sin(3x) + 4)^{\frac{3}{2}} + C$$

Another form of integral substitution involves identifying an 'annoying bit', which if simplified would enable the integral to be solved. Often this involves simplifying the argument of a square root or logarithm.

Example: substitute out the inconvenient component of the integral to solve:

$$\int (2x+1)\sqrt{x-3}\,dx$$

$$\text{let } u = x - 3$$

$$\frac{du}{dx} = 1$$

$$\int (2(u+3)+1)\sqrt{u}\frac{du}{dx}dx = \int (2u+7)\sqrt{u}\,du$$

$$= \int 2u^{\frac{3}{2}} + 7u^{\frac{1}{2}}\,du$$

$$= 2\left(\frac{2}{5}\right)u^{\frac{5}{2}} + 7\left(\frac{3}{2}\right)u^{\frac{3}{2}} + C$$

$$\int (2x+1)\sqrt{x-3}\,dx = \frac{4}{5}(x-3)^{\frac{5}{2}} + \frac{21}{2}(x-2)^{\frac{3}{2}} + C$$

Trigonometric identities

Integrals involving products of trigonometric functions can often be solved by using one or more trigonometric identities. There are two main cases to consider, depending on whether both functions have even powers or if at least one function has an odd power.

Odd case

In the odd case, one power can be split off to form a derivative, which can then be substituted out using standard substitution methods described above. A trigonometric identity is then used to eliminate the undesired function, so everything is written in terms of a single trig function. To see how this works, let n be odd, then we have:

$$\int \sin^m(x)\cos^n(x)\,dx = \int \sin^m(x)\cos^{n-1}(x)\cos(x)\,dx$$

We can now eliminate the $\cos(x)$ using the identity:

$$\cos^2(x) + \sin^2(x) = 1$$

Thus we have:

$$\int \sin^m(x)\cos^{n-1}(x)\cos(x)\,dx = \int \sin^m(x)\left(1-\sin^2(x)\right)^{\left(\frac{n-1}{2}\right)}\cos(x)\,dx$$

Now let $u = \sin(x)$, and hence $\frac{du}{dx} = \cos(x)$:

$$\int \sin^m(x)\left(1-\sin^2(x)\right)^{\left(\frac{n-1}{2}\right)}\cos(x)\,dx = \int u^m(1-u^2)^{\left(\frac{n-1}{2}\right)}du$$

The integrand on the right is now a polynomial, and so after expansion can be solved using standard methods. Hence we have for n being odd:

$$\int \sin^m(x)\cos^n(x)\,dx = \int u^m(1-u^2)^{\left(\frac{n-1}{2}\right)}\,du, \qquad \text{with } u = \sin(x)$$

Note that essentially the same process can be followed with integrals involving $\tan(x)$ and $\sec(x)$.

$$\tan^2(x) + 1 = \sec^2(x)$$

Even case

If both powers are even, the double angle identities are instead used:

$$\sin^2(x) = \frac{1}{2}(1-\cos(2x))$$
$$\cos^2(x) = \frac{1}{2}(1+\cos(2x))$$

Hence for m and n both even we have:

$$\begin{aligned}\int \sin^m(x)\cos^n(x)\,dx &= \int \left(\frac{1}{2}(1-\cos(2x))\right)^{\frac{m}{2}}\left(\frac{1}{2}(1+\cos(2x))\right)^{\frac{n}{2}}dx \\ &= \frac{1}{2^{(m+n)/2}}\int (1-\cos(2x))^{\frac{m}{2}}(1+\cos(2x))^{\frac{n}{2}}\,dx\end{aligned}$$

This process of substitution using the double angle identities continues until the right hand side is reduced to a sum of single powers of $\sin(nx)$ or $\cos(nx)$, where $n \in \mathbb{Z}$. Note that no derivative substitution is needed in this case.

Partial fractions

The method of partial fractions can be used to solve integrals of the form:

$$\int \frac{P^n(x)}{Q^m(x)}dx$$

Where $P^n(x)$ is a polynomial of degree $n < m$, and $Q^m(x)$ is a polynomial of degree m. Note that if the numerator has a higher degree than the denominator, polynomial long division can be performed prior to applying this technique.

The technique of partial fractions involves writing the integrand as a sum of smaller components, which can be individually integrated. In particular it turns out that for denominators that factorise into n distinct linear factors, it is always possible to write:

$$\frac{f(x)}{(x-a_1)(x-a_2)\dots(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}$$

In the other extreme case when there are n repeated factors in the denominator, we can write:

$$\frac{f(x)}{(x-a)^n} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_3}{(x-a)^n}$$

Intermediate cases in which there are some repeated factors and some distinct linear factors will factorise as a combination of these two extremes. To find the coefficients A_i, simply write the partial fraction equation as shown above, then perform cross multiplication on the right hand side until the denominators are equal. This will allow the numerators to be equated and hence the coefficients to be solved for.

Hence the integral for a function that has a denominator that can be factorised into n distinct linear factors is given by:

$$\int \frac{f(x)}{(x-a_1)(x-a_2)\dots(x-a_n)}dx = \sum_{i=1}^{n} A_i \ln|x-a_i| + C$$

In the other extreme case where there are n repeated factors in the denominator, we have:

$$\int \frac{f(x)}{(x-a)^n}dx = A_1 \ln|x-a| - \sum_{i=2}^{n} \frac{A_i}{(i-1)} \frac{1}{(x-a)^{i-1}} + C$$

Example: solve the following integral using partial fractions.

$$\begin{aligned}\int \frac{9x+1}{(x-3)(x+1)}dx &= \int \frac{A_1}{x-3} + \frac{A_2}{x+1}dx \\ &= \int \frac{A_1x + A_1}{(x-3)(x+1)} + \frac{A_2x - 3A_2}{(x-3)(x+1)}dx \\ &= \int \frac{(A_1+A_2)x + (A_1 - 3A_2)}{(x-3)(x+1)}dx\end{aligned}$$

Hence we have:

$$9 = A_1 + A_2 \text{ and } 1 = A_1 - 3A_2$$

Substituting we find:

$$\begin{aligned}A_1 &= 1 + 3A_2 \\ A_1 &= 1 + 3(9 - A_1) \\ A_1 &= 1 + 27 - 3A_1 \\ 4A_1 &= 28 \\ A_1 &= 7\end{aligned}$$

$$\begin{aligned}A_2 &= 9 - 7 \\ A_2 &= 9 - 7 \\ A_2 &= 2\end{aligned}$$

Thus we have:

$$\begin{aligned}\int \frac{9x+1}{(x-3)(x+1)}dx &= \int \frac{7}{x-3} + \frac{2}{x+1}dx \\ &= 7\int \frac{1}{x-3}dx + 2\int \frac{1}{x+1}dx \\ \int \frac{9x+1}{(x-3)(x+1)}dx &= 7\log|x-3| + 2\log|x+1| + C\end{aligned}$$

Example: solve the following integral using partial fractions.

$$\int \frac{2x-1}{x^2-6x+9}dx = \int \frac{2x-1}{(x-3)^2}dx$$

Because of the repeated factor we write as partial fractions as follows:

$$\frac{2x-1}{(x-3)^2} = \frac{A_1}{x-3} + \frac{A_2}{(x-3)^2}$$
$$= \frac{A_1 x - 3A_1}{(x-3)^2} + \frac{A_2}{(x-3)^2}$$

Hence we have:

$$2 = A_1 \text{ and } -1 = -3A_1 + A_2$$

Substituting we find:

$$-1 = -6 + A_2$$
$$A_2 = 5$$

$$\therefore \int \frac{2x-1}{(x-3)^2} dx = \int \frac{2}{x-3} + \frac{5}{(x-3)^2} dx$$
$$= 2\ln|x-3| - \frac{5}{x-3} + C$$

Trigonometric substitutions

A special subset of substitutions is very useful when the integrand takes the form of the sum or difference of perfect squares inside a square root. In such cases a trigonometric substitution can be performed, as indicated in the table below.

Integrand	Substitution
$\sqrt{a^2-x^2}$, $\frac{1}{\sqrt{a^2-x^2}}$	$x = a\sin(\theta), \frac{dx}{d\theta} = a\cos(\theta)$
$\sqrt{a^2+x^2}$, $\frac{1}{\sqrt{a^2+x^2}}$	$x = a\sinh(\theta), \frac{dx}{d\theta} = a\cosh(\theta)$
$\sqrt{x^2-a^2}$, $\frac{1}{\sqrt{x^2-a^2}}$	$x = a\cosh(\theta), \frac{dx}{d\theta} = a\sinh(\theta)$
$\frac{1}{a^2+x^2}$	$x = a\tan(\theta), \frac{dx}{d\theta} = a\sec^2(\theta)$

Example: solve the following integral using a trigonometric substitution.

$$\int \frac{1}{\sqrt{25+x^2}} dx = \int \frac{1}{\sqrt{5^2+(5\sinh(\theta))^2}} 5\cosh(\theta)\, d\theta$$
$$= \int \frac{5\cosh(\theta)}{5\sqrt{1+\sinh^2(\theta)}} d\theta$$
$$= \int \frac{5\cosh(\theta)}{5\cosh(\theta)} d\theta$$
$$= \int 1\, d\theta$$
$$= \theta + C$$

$$\int \frac{1}{\sqrt{25+x^2}}\,dx = \operatorname{arcsinh}\left(\frac{x}{5}\right) + C$$

Integration by parts

This technique can be used to solve integrals of the form:

$$\int g(x)f(x)\,dx$$

Where $g(x)$ can be differentiated easily, and $f(x)$ can be integrated easily. The technique can be derived using the product rule for differentiation.

$$\begin{aligned}
\frac{d}{dx}\big(f(x)g(x)\big) &= f'(x)g(x) + f(x)g'(x) \\
\int \frac{d}{dx}\big(f(x)g(x)\big)\,dx &= \int f'(x)g(x) + f(x)g'(x)\,dx \\
f(x)g(x) &= \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx \\
\int f'(x)g(x)\,dx &= f(x)g(x) - \int f(x)g'(x)\,dx
\end{aligned}$$

Example: solve the following integral using integration by parts.

$$\begin{aligned}
\int xe^{5x}\,dx &= (x)\left(\frac{1}{5}e^{5x}\right) - \int \left(\frac{1}{5}e^{5x}\right)(1)dx \\
&= \frac{x}{5}e^{5x} - \frac{1}{5}\int e^{5x}\,dx \\
&= \frac{x}{5}e^{5x} - \frac{1}{25}e^{5x} + C \\
\int xe^{5x}\,dx &= \frac{e^{5x}}{5}\left(x - \frac{1}{5}\right) + C
\end{aligned}$$

In cases involving a trigonometric function, performing integration by parts twice will recover the original form of the function (since the second derivative of $\sin x$ is $-\sin x$), thereby enabling the original integral to be solved by rearranging.

Example: solve the following integral using recursive integration by parts.

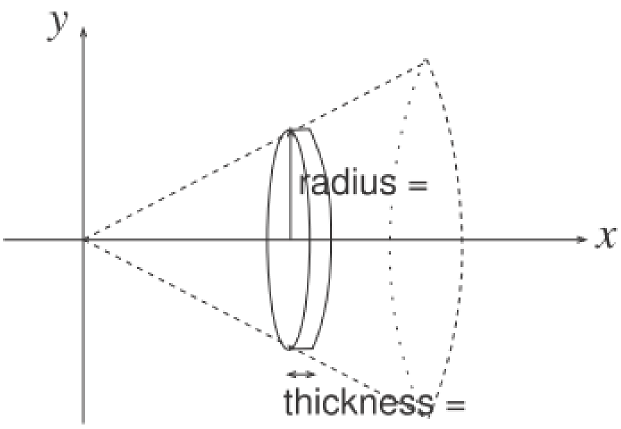
$$\begin{aligned}
\int e^{3x}\sin(2x)\,dx &= e^{3x}\left(-\frac{1}{2}\cos(2x)\right) - \int 3e^{3x}\left(-\frac{1}{2}\cos(2x)\right)dx \\
&= -\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{2}\int e^{3x}\cos(2x)\,dx \\
&= -\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{2}\left[e^{3x}\left(\frac{1}{2}\sin(2x)\right) - \int 3e^{3x}\left(\frac{1}{2}\sin(2x)\right)dx\right] \\
&= -\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{2}\left[\frac{1}{2}e^{3x}\sin(2x) - \frac{3}{2}\int e^{3x}\sin(2x)\,dx\right] \\
\int e^{3x}\sin(2x)\,dx &= -\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{4}e^{3x}\sin(2x) - \frac{9}{4}\int e^{3x}\sin(2x)\,dx \\
\frac{13}{4}\int e^{3x}\sin(2x)\,dx &= -\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{4}e^{3x}\sin(2x) \\
\int e^{3x}\sin(2x)\,dx &= -\frac{2}{13}e^{3x}\cos(2x) + \frac{3}{13}e^{3x}\sin(2x) + C
\end{aligned}$$

Volumes of solids of revolution

Rotation about the x-axis

When rotating a function about the x-axis, the thickness is taken as an infinitesimal, with the radius written as a function of x, which is then integrated over the domain.

$$V_{\text{slice}} = \pi(\text{radius})^2 \cdot \text{thickness} .$$



$$V_{slice} = \int_0^h \pi[r(x)]^2 \, dx$$

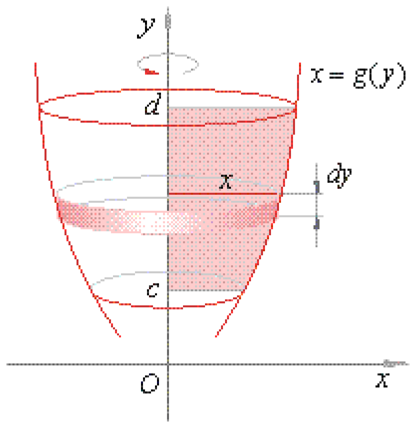
In the simple case of a cone 45-degree cone where $r = x$ we find:

$$V_{slice} = \int_0^h \pi x^2 \, dx = \frac{1}{3}\pi h^3$$

This is the formula for the volume of a 45-degree cone.

Rotation about the y-axis

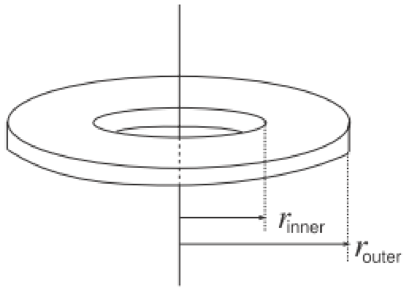
The method is effectively the same when considering rotation about the y-axis, except in this case the radius is written as a function of y, and the integration is performed over y.



$$V_{slice} = \int_0^w \pi[x(y)]^2 \, dy$$

Volume by washers

As shown in the diagram below, the same method can also be applied in the case of hollow regions.



The volume of a washer is:

$$\begin{aligned} V_{\text{washer}} &= V_{\text{large disc}} - V_{\text{small disc}} \\ &= \pi \cdot r_{\text{outer}}^2 \cdot \text{thickness} - \pi \cdot r_{\text{inner}}^2 \cdot \text{thickness} \\ &= \pi \cdot \left(r_{\text{outer}}^2 - r_{\text{inner}}^2\right) \cdot \text{thickness} \end{aligned}$$

Differential Equations

Introduction to differential equations

A ordinary differential equation is an equation that involves x, y, and derivatives of y such as dy/dx or higher derivatives. The order of a differential equation is the highest derivative found in the equation. Most commonly only first and second order differential equations are used.

The general solution of a differential equation defines the complete set of all solutions for that equation. A particular solution is one element from the general solutions that satisfies more specific conditions.

Solution by direct integration

The simplest form of differential equation has a right hand side that is only a function of x. These equations can be solved directly by simple integration.

$$\frac{dy}{dx} = f(x)$$
$$y(x) = \int f(x)\,dx$$

Example: solve the following using direct integration.

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$
$$y = \int \frac{1}{1+x^2}\,dx$$

$$\text{let } x = \tan(\theta), \frac{dx}{d\theta} = \sec^2(\theta)$$
$$y = \int \frac{1}{1+\tan^2(\theta)} \sec^2(\theta)\,d\theta$$
$$y = \int \frac{\sec^2(\theta)}{1+\tan^2(\theta)}\,d\theta$$
$$y = \int 1\,d\theta$$
$$y = \theta + C$$
$$y = \arctan(x) + C$$

A slightly more complex form occurs when the right hand side is only a function of y. These equations can also be solved by direct integration after some simple rearrangement.

$$\frac{dy}{dx} = kg(y)$$
$$\frac{1}{g(y)}dy = kdx$$
$$\int \frac{1}{g(y)}dy = \int k\,dx$$

Example: solve the following using direct integration

$$\frac{dy}{dx} = \sqrt{4-y^2}$$

$$\frac{1}{\sqrt{4-y^2}}dy = dx$$

$$\int \frac{1}{\sqrt{4-y^2}}dy = \int 1\,dx$$

$$\text{let } y = 2\sin(\theta)$$

$$x = \arcsin\left(\frac{y}{2}\right) + C$$

$$x = \int 1\,d\theta$$

$$x = \theta + C$$

$$x = \arcsin\left(\frac{y}{2}\right) + C$$

$$\frac{y}{2} = \sin(x - C)$$

$$y = 2\sin(x - C)$$

Separable differential equations

A more complex type of differential equation again occurs when the right hand side can be factorised into a function of x only and a function of y only. This type of equation is called a separable differential equation.

$$\frac{dy}{dx} = f(x)g(y)$$

These equations can be solved by separation of variables.

$$\frac{1}{g(y)}dy = f(x)dx$$

$$\int \frac{1}{g(y)}dy = \int f(x)\,dx$$

Example: solve the following separable differential equation

$$\frac{dy}{dx} = \frac{1}{2y\sqrt{1-x^2}}, \text{with IV } (x,y) = (0,3)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{2y}$$

$$2y\,dy = \frac{1}{\sqrt{1-x^2}}dx$$

$$2\int y\,dy = \int \frac{1}{\sqrt{1-x^2}}dx$$

$$\text{let } x = \sin(\theta)$$

$$y^2 = \int \frac{1}{\sqrt{1-\sin^2(\theta)}}\cos(\theta)\,d\theta$$

$$y^2 = \int 1\,d\theta$$

$$y^2 = \theta + C$$

$$y^2 = \arcsin(x) + C$$

$$(3)^2 = \arcsin(0) + C$$

$$C = 9$$

$$\therefore y = \sqrt{\arcsin(x) + 9}$$

Example: the logistic model can describe population growth rates in the presence of competition (negative x^2 term) and harvesting (removing a certain number of the population per time increment, negative constant h). This equation takes the form:

$$\frac{dx}{dt} = kx - \frac{k}{a}x^2 - h$$
$$\frac{dx}{dt} = kx\left(1 - \frac{x}{a}\right) - h$$

Without the harvesting term this differential equation becomes separable and can be solved exactly.

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{a}\right)$$
$$\frac{1}{kx\left(1 - \frac{x}{a}\right)}\frac{dx}{dt} = 1$$
$$\int \frac{1}{kx\left(1 - \frac{x}{a}\right)} dx = \int dt$$
$$\frac{1}{k}\int \frac{a}{x(a-x)} dx = t + C$$
$$\frac{a}{k}\int \frac{1}{x(a-x)} dx = t + C$$

Solve by partial fractions:

$$\frac{1}{x(a-x)} = \frac{A}{x} + \frac{B}{a-x}$$
$$\frac{1}{x(a-x)} = \frac{A(a-x)}{x(a-x)} + \frac{Bx}{x(a-x)}$$
$$1 = A(a-x) + Bx$$
$$1 = Aa - Ax + Bx$$
$$1 = Aa + x(B - A)$$
$$\therefore A = \frac{1}{a}, \qquad B = \frac{1}{a}$$

Solving the integral yields:

$$\frac{a}{k}\int \frac{1}{ax} + \frac{1}{a(a-x)} dx = t + C$$
$$\frac{1}{k}\int \frac{1}{x} + \frac{1}{(a-x)} dx = t + C$$
$$\frac{1}{k}[\ln(x) - \ln(a-x)] = t + C$$
$$\ln\left(\frac{x}{a-x}\right) = kt + C$$
$$\frac{x}{a-x} = \exp(kt + C)$$
$$\frac{x}{a-x} = \exp(C)\exp(kt)$$
$$\frac{x}{a-x} = A\exp(kt)$$

$$x = aA\exp(kt) - A\exp(kt)\,x$$
$$x + A\exp(kt)\,x = aA\exp(kt)$$
$$x(1 + A\exp(kt)) = aA\exp(kt)$$
$$x = \frac{aA\exp(kt)}{1 + A\exp(kt)}$$
$$x = \frac{a}{1 + A\exp(-kt)}$$

To solve for A we can write $x(t) = x_0$ and then we have:

$$x_0 = \frac{a}{1 + A\exp(0)}$$
$$x_0(1 + A) = a$$
$$x_0 + Ax_0 = a$$
$$A = \frac{a - x_0}{x_0}$$

Hence we find the overall solution:

$$\boxed{x = \frac{a}{1 + \left(\frac{a - x_0}{x_0}\right)\exp(-kt)}}$$

Solving for the steady-state solution when the derivative is equal to zero:

$$0 = kx\left(1 - \frac{x}{a}\right)$$
$$0 = x(a - x)$$
$$x = 0 \ \text{ or } \ x = a$$

This is consistent with the fact that $\exp(-kt) \to 0$ as $t \to \infty$, and hence x tends to a.

First-order linear differential equations

A first-order linear differential equation has the form:

$$\frac{dy}{dx} + f(x)y = g(x)$$

This equation cannot be solved by direct integration. Instead, we solve this by using the technique called an integrating factor. This requires us to find a function $v(x)$ that satisfies the following condition:

$$\frac{d\big(v(x)\big)}{dx} = f(x)v(x)$$
$$\frac{1}{v(x)}dv(x) = f(x)dx$$
$$\int \frac{1}{v(x)}v(x) = \int f(x)\,dx$$
$$\ln\big(v(x)\big) = \int f(x)\,dx$$
$$v(x) = \exp\left(\int f(x)\,dx\right)$$

The use of such a function is that we can now rewrite the original differential equation in terms of the derivative of $v(x)y$ as follows:

$$
\begin{aligned}
\frac{dy}{dx} + f(x)y &= g(x) \\
v(x)\frac{dy}{dx} + v(x)f(x)y &= v(x)g(x) \\
\frac{d}{dx}(v(x)y) &= v(x)g(x) \\
v(x)y &= \int v(x)g(x)\,dx \\
y &= \frac{1}{v(x)}\int v(x)g(x)\,dx
\end{aligned}
$$

The function $v(x) = \exp(\int f(x)\,dx)$ is called the integrating factor. It allows us to solve the equation by multiplying both sides by $v(x)$, then combining the two left hand side terms into a single derivative with respect to x, then integrating.

Example: the sum of the voltages in a circuit with a periodic voltage source, an inductor, and a resistor (an RL circuit) is given by the equation:

$$0 = V_0\cos(\phi t) - L\frac{dI}{dt} - RI$$

Where L is the inductance and R is the resistance, with current I. This can be written as:

$$
\begin{aligned}
L\frac{dI}{dt} &= V_0\cos(\phi t) - RI \\
\frac{dI}{dt} + \frac{R}{L}I &= \frac{V_0}{L}\cos(\phi t)
\end{aligned}
$$

Note that this has the form of a linear differential equation. We first find the integrating factor:

$$
\begin{aligned}
v(t) &= \exp\left(\int f(t)\,dt\right) \\
&= \exp\left(\int \frac{R}{L}\,dt\right) \\
v(t) &= \exp\left(\frac{R}{L}t\right)
\end{aligned}
$$

Using the result from above we have:

$$
\begin{aligned}
I(t) &= \frac{1}{v(t)}\int v(t)g(t)\,dt \\
&= \frac{1}{\exp\left(\frac{R}{L}t\right)}\int \exp\left(\frac{R}{L}t\right)\frac{V_0}{L}\cos(\phi t)\,dt \\
I(t) &= \frac{V_0/L}{\exp\left(\frac{R}{L}t\right)}\int \exp\left(\frac{R}{L}t\right)\cos(\phi t)\,dt
\end{aligned}
$$

To solve this integral we can use repeated integration by parts:

$$\int \exp\left(\frac{R}{L}t\right)\cos(\phi t)\,dt = \frac{L}{R}\exp\left(\frac{R}{L}t\right)\cos(\phi t) - \int \frac{L}{R}\exp\left(\frac{R}{L}t\right)(-\phi)\sin(\phi t)\,dt$$

$$= \frac{L}{R}\exp\left(\frac{R}{L}t\right)\cos(\phi t) + \frac{L}{R}\phi\int \exp\left(\frac{R}{L}t\right)\sin(\phi t)\,dt$$

$$= \frac{L}{R}\exp\left(\frac{R}{L}t\right)\cos(\phi t) + \frac{L}{R}\phi\left[\frac{L}{R}\exp\left(\frac{R}{L}t\right)\sin(\phi t) - \int \frac{L}{R}\exp\left(\frac{R}{L}t\right)\phi\cos(\phi t)\,dt\right]$$

$$= \frac{L}{R}\exp\left(\frac{R}{L}t\right)\cos(\phi t) + \phi\left(\frac{L}{R}\right)^2\exp\left(\frac{R}{L}t\right)\sin(\phi t) - \left(\frac{L}{R}\right)^2\phi^2\int \exp\left(\frac{R}{L}t\right)\cos(\phi t)\,dt$$

$$\left(1+\left(\frac{L}{R}\right)^2\phi^2\right)\int \exp\left(\frac{R}{L}t\right)\cos(\phi t)\,dt = \frac{L}{R}\exp\left(\frac{R}{L}t\right)\cos(\phi t) + \phi\left(\frac{L}{R}\right)^2\exp\left(\frac{R}{L}t\right)\sin(\phi t)$$

$$\int \exp\left(\frac{R}{L}t\right)\cos(\phi t)\,dt = \frac{\frac{L}{R}\exp\left(\frac{R}{L}t\right)\cos(\phi t) + \phi\left(\frac{L}{R}\right)^2\exp\left(\frac{R}{L}t\right)\sin(\phi t)}{\left(1+\left(\frac{L}{R}\right)^2\phi^2\right)}$$

$$\int \exp\left(\frac{R}{L}t\right)\cos(\phi t)\,dt = \exp\left(\frac{R}{L}t\right)\left[\frac{\left(\frac{L}{R}\right)\cos(\phi t) + \phi\left(\frac{L}{R}\right)^2\sin(\phi t)}{\left(1+\left(\frac{L}{R}\right)^2\phi^2\right)}\right] + C$$

Substituting this into the solution we find:

$$I(t) = \frac{\frac{V_0}{L}}{\exp\left(\frac{R}{L}t\right)}\exp\left(\frac{R}{L}t\right)\left[\frac{\left(\frac{L}{R}\right)\cos(\phi t) + \phi\left(\frac{L}{R}\right)^2\sin(\phi t)}{\left(1+\left(\frac{L}{R}\right)^2\phi^2\right)}\right] + \frac{\frac{V_0 C}{L}}{\exp\left(\frac{R}{L}t\right)}$$

$$I(t) = \frac{\frac{V_0}{L}}{1+\left(\frac{L}{R}\right)^2\phi^2}\left(\left(\frac{L}{R}\right)\cos(\phi t) + \phi\left(\frac{L}{R}\right)^2\sin(\phi t)\right) + \frac{\frac{V_0 C}{L}}{\exp\left(\frac{R}{L}t\right)}$$

$$I(t) = \frac{V_0}{L\left(\frac{R}{L}\right)^2\left(1+\left(\frac{L}{R}\right)^2\phi^2\right)}\left(\left(\frac{R}{L}\right)\cos(\phi t) + \phi\sin(\phi t)\right) + \frac{\frac{V_0 C}{L}}{\exp\left(\frac{R}{L}t\right)}I(t)$$

$$I(t) = \frac{V_0}{L\left(\left(\frac{R}{L}\right)^2+\phi^2\right)}\left(\left(\frac{R}{L}\right)\cos(\phi t) + \phi\sin(\phi t)\right) + \frac{V_0 C}{L}\exp\left(-\frac{R}{L}t\right)$$

To solve for C we substitute $I(t_0) = I_0$ for $t_0 = 0$:

$$I_0 L = \frac{V_0}{L\left(\left(\frac{R}{L}\right)^2+\phi^2\right)}\left(\left(\frac{R}{L}\right)\right) + \frac{V_0 C}{L}$$

$$I_0 L = \frac{V_0}{\left(\left(\frac{R}{L}\right)^2+\phi^2\right)}\left(\frac{R}{L}\right) + V_0 C$$

$$CV_0 = I_0 L - \frac{V_0}{\left(\left(\frac{R}{L}\right)^2+\phi^2\right)}\left(\frac{R}{L}\right)$$

$$C = \frac{I_0 L}{V_0} - \frac{\left({}^{R}/_{L}\right)}{\left(\left({}^{R}/_{L}\right)^2+\phi^2\right)}$$

Hence we arrive at:

$$I(t) = \frac{V_0}{L\left(\left(\frac{R}{L}\right)^2 + \phi^2\right)}\left(\left(\frac{R}{L}\right)\cos(\phi t) + \phi\sin(\phi t)\right) + \frac{V_0}{L}\exp\left(-\frac{R}{L}t\right)\left(\frac{I_0 L}{V_0} - \frac{\left({}^{R}/_{L}\right)}{\left(\left({}^{R}/_{L}\right)^2 + \phi^2\right)}\right)$$

$$= \frac{V_0}{L\left(\left(\frac{R}{L}\right)^2 + \phi^2\right)}\left(\left(\frac{R}{L}\right)\cos(\phi t) + \phi\sin(\phi t)\right) + I_0\exp\left(-\frac{R}{L}t\right) - \frac{V_0\left(\frac{R}{L}\right)}{L\left(\left(\frac{R}{L}\right)^2 + \phi^2\right)}\exp\left(-\frac{R}{L}t\right)$$

$$\boxed{I(t) = I_0\exp\left(-\frac{R}{L}t\right) - \frac{V_0\left(\frac{R}{L}\right)}{L\left(\left(\frac{R}{L}\right)^2 + \phi^2\right)}\exp\left(-\frac{R}{L}t\right) + \frac{V_0}{L\left(\left(\frac{R}{L}\right)^2 + \phi^2\right)}\left(\left(\frac{R}{L}\right)\cos(\phi t) + \phi\sin(\phi t)\right)}$$

Second order linear differential equations

Second order differential equations contain second derivatives in addition to (or instead of) first derivatives. Many such equations are difficult to solve, but simple methods can be used to solve a special class of such equations known as linear differential equations with constant coefficients.

Homogenous equations

These have the form:

$$a\left(\frac{dy^2}{dx^2}\right) + b\left(\frac{dy}{dx}\right) + cy = 0$$

They can also be written in more compact notation:

$$ay'' + by' + cy = 0$$

Solutions of this equation always take the form:

$$y(x) = e^{\lambda x}$$

Substituting in this particular solution we find:

$$a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0$$
$$(a\lambda^2 + b\lambda + c)e^{\lambda x} = 0$$
$$a\lambda^2 + b\lambda + c = 0$$

This is known as the characteristic equation. The solutions to this equation for λ determine the form of the general solution of the differential equation under consideration. There are three possible cases.

Case	Description	Values of λ	General Solution
$b^2 - 4ac > 0$	Two distinct values	$\lambda_1, \lambda_2 \in \mathbb{R}$	$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
$b^2 - 4ac = 0$	Single solution	λ	$y(x) = Ae^{\lambda x} + Bxe^{\lambda x}$
$b^2 - 4ac < 0$	Two complex conjugate values	$\lambda_1 = \alpha + i\beta, \lambda = \alpha - i\beta$	$y(x) = Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x}$

Example: solve the following homogenous linear differential equation if $y(0) = -1$ and $y'(0) = 2$.

$$y'' - 4y' + 13y = 0$$

Use the trial solution $y = e^{\lambda x}$ we have:

$$\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} + 13e^{\lambda x} = 0$$
$$(\lambda^2 - 4\lambda + 13)e^{\lambda x} = 0$$
$$\lambda^2 - 4\lambda + 13 = 0$$
$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\lambda = \frac{4 \pm \sqrt{16 - 4(1)(13)}}{2(1)}$$
$$\lambda = 2 \pm \frac{1}{2}\sqrt{-36}$$
$$\lambda = 2 \pm \frac{1}{2}6i$$
$$\lambda = 2 \pm 3i$$

We therefore find the solution (with complex coefficients):

$$y(x) = Ae^{(2+3i)x} + Be^{(2-3i)x}$$
$$y(x) = (A_1 + iA_2)e^{(2+3i)x} + (B_1 + iB_2)e^{(2-3i)x}$$

The coefficients can be found using the initial conditions:

$$y(0) = (A_1 + iA_2)e^{(2+3i)(0)} + (B_1 + iB_2)e^{(2-3i)(0)}$$
$$y(0) = (A_1 + iA_2)e^{(2+3i)(0)} + (B_1 + iB_2)e^{(2-3i)(0)}$$
$$-1 = (A_1 + iA_2) + (B_1 + iB_2)$$
$$\therefore B_2 = -A_2$$
$$\therefore B_1 = -1 - A_1$$

$$y'(x) = (2 + 3i)Ae^{(2+3i)x} + (2 - 3i)Be^{(2-3i)x}$$
$$y'(0) = (2 + 3i)Ae^{(2+3i)(0)} + (2 - 3i)Be^{(2-3i)(0)}$$
$$2 = (2 + 3i)(A_1 + iA_2) + (2 - 3i)(B_1 + iB_2)$$
$$2 = (2 + 3i)(A_1 + iA_2) + (2 - 3i)(-1 - A_1 - iA_2)$$
$$2 = 2A_1 + 2iA_2 + 3iA_1 + 3i^2A_2 - 2 - 2A_1 - i2A_2 + 3i + 3iA_1 + 3i^2A_2$$
$$4 = 6iA_1 - 6A_2 + 3i$$
$$4 + 6A_2 = (6A_1 + 3)i$$
$$\therefore A_2 = -\frac{2}{3}, A_1 = -\frac{1}{2}$$

Substituting these back into the solution equation we have:

$$y(x) = \left(-\frac{1}{2} - \frac{2}{3}i\right)e^{(2+3i)x} + \left(-\frac{1}{2} + \frac{2}{3}i\right)e^{(2-3i)x}$$
$$y(x) = \left(-\frac{1}{2} - \frac{2}{3}i\right)e^{2x}(\cos(3x) + i\sin(3x)) + \left(-\frac{1}{2} + \frac{2}{3}i\right)e^{2x}(\cos(3x) - i\sin(3x))$$

$$y(x) = e^{2x}\left(-\frac{1}{2}\cos(3x) - \frac{2}{3}\cos(3x)\,i - \frac{1}{2}\sin(3x)\,i - \frac{2}{3}(i^2)\sin(3x)\right)$$
$$+ e^{2x}\left(-\frac{1}{2}\cos(3x) + \frac{2}{3}\cos(3x)\,i + \frac{1}{2}\sin(3x)\,i - \frac{2}{3}(i^2)\sin(3x)\right)$$

At last we find the full general solution:

$$\boxed{y(x) = -e^{2x}\cos(3x) + \frac{4}{3}e^{2x}\sin(3x)}$$

Inhomogeneous equations

These have the more general form:

$$a\left(\frac{dy^2}{dx^2}\right) + b\left(\frac{dy}{dx}\right) + cy = f(x)$$

Or equivalently:

$$ay'' + by' + cy = f(x)$$

Note the key difference from homogenous equations being the presence of a function of x on the right-hand side.

To solve these types of equations, we first find the solution to the homogenous version of the same equation ($y_h(x)$), then find a particular solution ($y_p(x)$) to the inhomogeneous equation. The general solution to the inhomogeneous equation will then be:

$$y(x) = y_h(x) + y_p(x)$$

Particular solutions can be found by substituting various trial-solutions into the equation and solving for any unknown coefficients. The following table indicates which trial solutions should be used.

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n\ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k\cos\omega x$	$K\cos\omega x + M\sin\omega x$
$k\sin\omega x$	
$ke^{\alpha x}\cos\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$
$ke^{\alpha x}\sin\omega x$	

Example: solve the following inhomogeneous differential equation.

$$y'' + 2y' - 8y = 3 - 24x^2 + 7e^{3x}$$

First solve the inhomogeneous equation.

$$y'' + 2y' - 8y = 0$$

$$\lambda^2 + 2\lambda - 8 = 0$$
$$\lambda = \frac{-2 \pm \sqrt{4 + 32}}{2}$$
$$\lambda = -1 \pm 3$$
$$\lambda = -4, 2$$

This yields the homogenous solution:

$$y_h(x) = Ae^{-4x} + Be^{2x}$$

We now find a particular solution using the trial solution $y_p(x) = Ax^2 + Bx + C + De^{2x}$

$$y_p'(x) = 2Ax + B + 2De^{2x}$$
$$y_p''(x) = 2A + 4De^{2x}$$

Substituting into the inhomogeneous equation to find coefficients:

$$y'' + 2y' - 8y = 3 - 24x^2 + 7e^{3x}$$
$$2A + 9De^{3x} + 2(2Ax + B + 3De^{3x}) - 8(Ax^2 + Bx + C + De^{3x}) = 3 - 24x^2 + 7e^{3x}$$
$$2A + 9De^{3x} + 4Ax + 2B + 6De^{3x} - 8Ax^2 - 8Bx - 8C - 8De^{3x} = 3 - 24x^2 + 7e^{3x}$$
$$(2A + 2B - 8C) + (4A - 8B)x - 8Ax^2 + (9D - 8D + 6D)e^{3x} = 3 - 24x^2 + 7e^{3x}$$

Solving for the coefficients:

$$-8A = -24$$
$$A = 3$$

$$4A - 8B = 0$$
$$A - 2B = 0$$
$$2B = 3$$
$$B = \frac{3}{2}$$

$$2A + 2B - 8C = 3$$
$$6 + 3 - 8C = 3$$
$$8C = 6$$
$$C = \frac{3}{4}$$

$$9D - 8D + 6D = 7$$
$$7D = 7$$
$$D = 1$$

This yields the particular solution:

$$y_p(x) = 3x^2 + \frac{3}{2}x + \frac{3}{4} + e^{2x}$$

We therefore find the general solution as the sum of homogeneous and particular solutions:

$$y(x) = y_h(x) + y_p(x)$$
$$y(x) = Ae^{-4x} + Be^{2x} + 3x^2 + \frac{3}{2}x + \frac{3}{4} + e^{2x}$$

Example: the equation for an RLC circuit with a resistor with resistance R, an inductor of inductance L, and a capacitor of capacitance C, is given by:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0$$
$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{CL}I = 0$$

The characteristic equation is:

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{CL} = 0$$
$$\lambda = \frac{-\left(\frac{R}{L}\right) \pm \sqrt{\left(\frac{R}{L}\right)^2 - 4\left(\frac{1}{CL}\right)}}{2}$$
$$\lambda = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}$$
$$\lambda = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

If we define a new set of variables $\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$ then we have:

$$\lambda = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

If $\alpha < \omega_0$ then the system is said to be overdamped. The result is a set of decaying solutions:

$$\boxed{I(t) = Ae^{\left(-\alpha+\sqrt{\alpha^2-\omega_0^2}\right)t} + Be^{\left(-\alpha-\sqrt{\alpha^2-\omega_0^2}\right)t}}$$

If $\alpha > \omega_0$ then the system is said to be underdamped. The result is a set of oscilating and decaying solutions:

$$I(t) = Ae^{\left(-\alpha+i\sqrt{\alpha^2-\omega_0^2}\right)t} + Be^{\left(-\alpha-i\sqrt{\alpha^2-\omega_0^2}\right)t}$$
$$= Ae^{-\alpha t}\left(\cos\left(\sqrt{\alpha^2-\omega_0^2}t\right) + i\sin\left(\sqrt{\alpha^2-\omega_0^2}t\right)\right)$$
$$+ Be^{-\alpha t}\left(\cos\left(\sqrt{\alpha^2-\omega_0^2}t\right) - i\sin\left(\sqrt{\alpha^2-\omega_0^2}t\right)\right)$$
$$= (A+B)e^{-\alpha t}\cos\left(\sqrt{\alpha^2-\omega_0^2}t\right) + (A-B)ie^{-\alpha t}\sin\left(\sqrt{\alpha^2-\omega_0^2}t\right)$$
$$\boxed{I(t) = C_1e^{-\alpha t}\cos\left(\sqrt{\alpha^2-\omega_0^2}t\right) + C_2e^{-\alpha t}\sin\left(\sqrt{\alpha^2-\omega_0^2}t\right)}$$

If $\alpha = \omega_0$ then the system is said to be critically damped. The result is a set of rapidly decaying solutions:

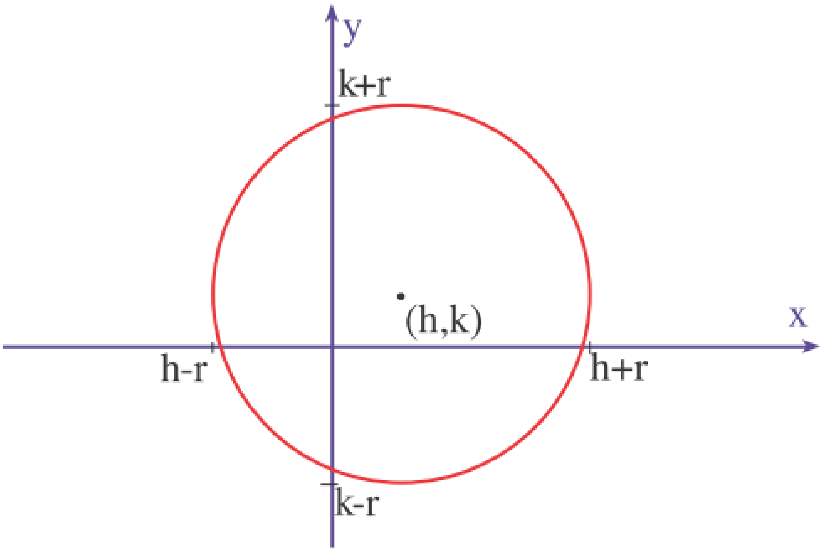
$$\boxed{I(t) = D_1e^{-\alpha t} + D_2te^{-\alpha t}}$$

Multivariate Calculus

Graphs of conic sections

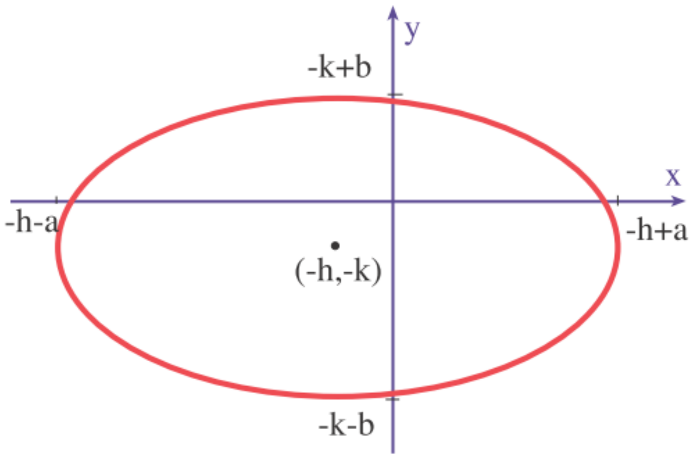
While a circle cannot be represented by a single function, the general formula for a circle is centered at (h, k) and radius r is given by:

$$(x-h)^2+(x-k)^2=r^2$$



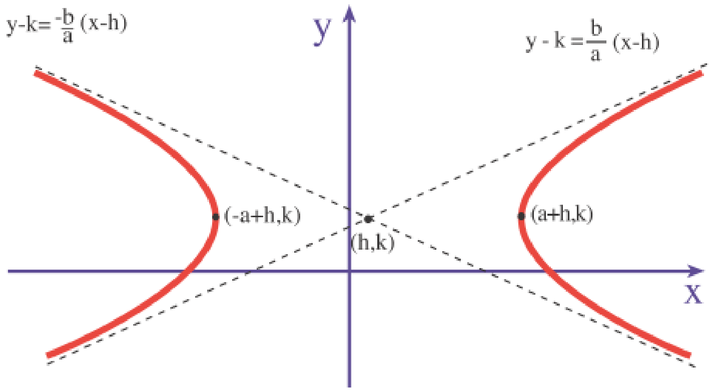
This can be adapted into an equation for an ellipse by introducing separate dilation factors for each dimension. Note that an ellipse does not have a radius, but rather has a major axis and a minor axis.

$$\frac{(x-h)^2}{a^2}+\frac{(x-k)^2}{b^2}=1$$



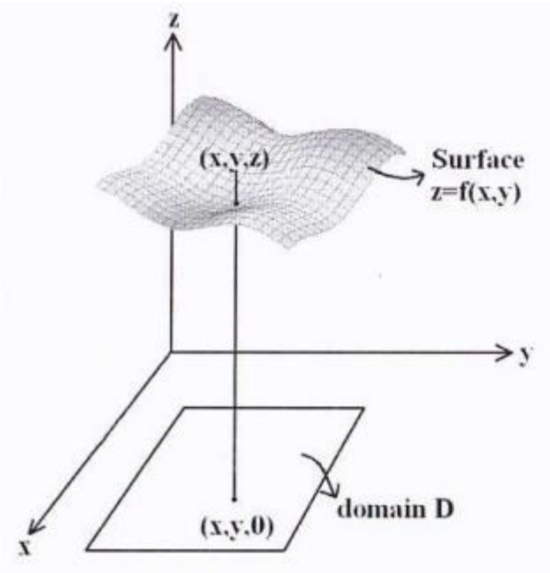
A modification with a negative sign gives rise to the equation for a hyperbola:

$$\frac{(x-h)^2}{a^2}-\frac{(x-k)^2}{b^2}=1$$



Functions of two variables

A function of two variables is a generalization of the concept of a function of one variable. It is a mapping that assigns a single number to each pair of real numbers (x, y) in some subset of the two-dimensional domain $\mathbb{R}^2$. Such functions can be depicted in a three-dimensional graph.



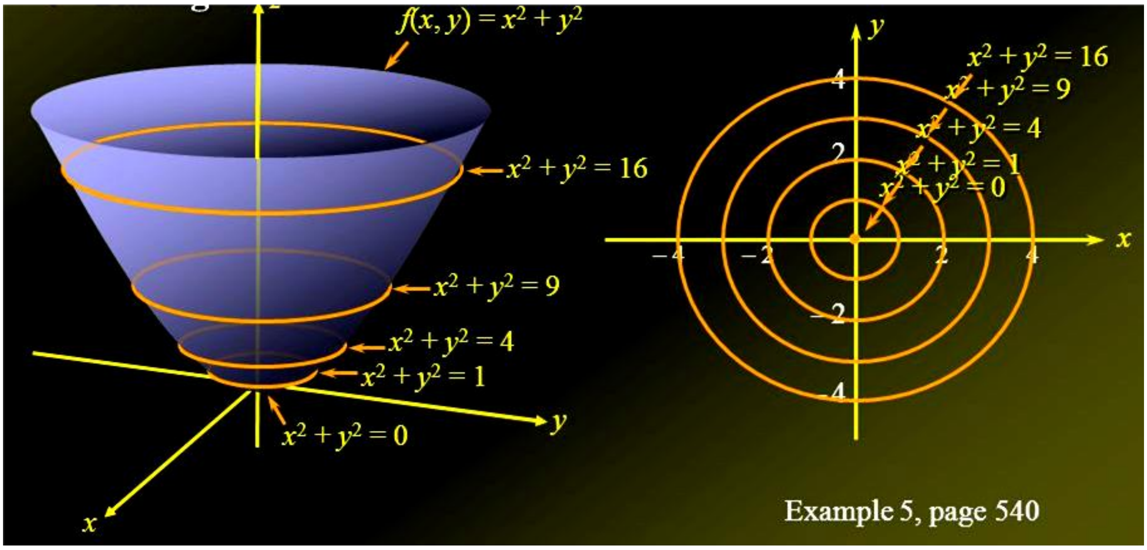
As an example of a two-dimensional function, any plane in $\mathbb{R}^3$ can be expressed as:

$$ax + by + cz = d$$
$$cz = d - ax - by$$
$$z = \frac{d}{c}$$
$$f(x, y) = -\frac{a}{c}x - \frac{b}{c}y + \frac{d}{c}$$

Level curves

A curve on the surface $z = f(x, y)$ for which z is a constant is called a contour, or when represented in the xy-plane is called a level curve. It is formally defined as:

$$\{(x, y): f(x, y) = c\}$$



Partial derivatives

The limit of a two-variable function is defined as:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

If when (x,y) approaches (x_0,y_0) along any possible path in the domain, $f(x,y)$ gets arbitrarily close to L.

Such a function is said to be continuous at $(x,y) = (x_0,y_0)$ if:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

A partial derivative measures the rate of change of f when one variable changes while holding the other variable constant. The two partial derivates for a two-variable function are:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\left[\frac{f(x+h,y)-f(x,y)}{h}\right]$$
$$\frac{\partial f}{\partial y} = \lim_{h\to 0}\left[\frac{f(x,y+h)-f(x,y)}{h}\right]$$

Partial derivatives are computed in exactly the same way as regular derivatives, except that the other variable not being differentiated with respect to is treated as a constant.

Example: find both partial derivatives for the following function.

$$f(x,y) = 3x^3y^2 + 3xy^4$$
$$\frac{\partial f}{\partial x} = 9x^2y^2 + 3y^4$$
$$\frac{\partial f}{\partial y} = 6x^3y + 12xy^3$$

Second order partial derivatives are defined similarly to first order partial derivatives. Note the variety of notations for such derivatives.

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$
$$\frac{\partial^2 f}{\partial y \partial x} = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

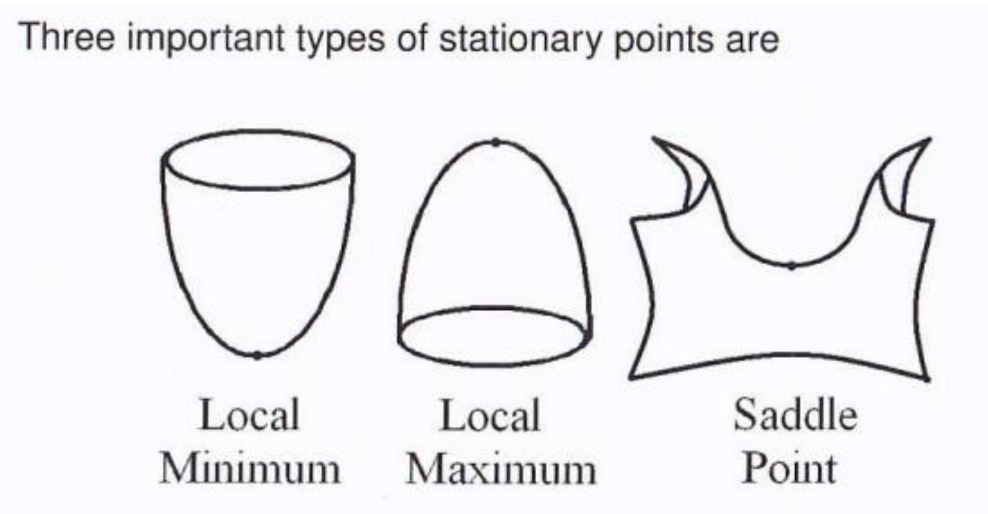
If the second order partial derivatives exist and are continuous, then $f_{xy} = f_{yx}$.

The value of a function near a point (x_0, y_0) can be approximated using its partial derivatives:

$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}|_{(x_0,y_0)}(y - y_0)$$

Stationary points

A stationary point of $f(x,y)$ is a point where both partial derivatives are equal to zero. This corresponds to a point at which the tangent plane to the graph is parallel to the xy-plane.



If all partial derivatives exist in an open disk around (x_0, y_0), we can construct what is called the Hessian matrix, defined as:

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

The determinate of the Hessian matrix is:

$$\det H = f_{xx} f_{yy} - f_{xy} f_{yx}$$
$$\det H = f_{xx} f_{yy} - f_{xy}^2$$

If $\det H > 0$ we say the matrix is positive definite. This is useful in identifying the nature of a stationary point as follows:

Condition	**Type of stationary point**
$\det H > 0$ and $f_{xx} < 0$ or $f_{yy} < 0$	Local minimum
$\det H > 0$ and $f_{xx} < 0$ or $f_{yy} > 0$	Local maximum
$\det H < 0$	Saddle point
$\det H = 0$	Test inconclusive

Double integrals

Integration in functions of two variables works in essentially the same way as in functions of one variable. Partial integrals involve integrating with respect to only one variable, keeping the other constant.

Example: solve the following integration.

$$\int 3x^2y + 12y^2x^3\,dx = x^3y + 3y^2x^4 + c(y)$$

Functions of two variables can also be integrated with respect to both variables, forming what is called a double integral defined over some area A and domain D.

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dxdy$$

This integral is interpreted as the volume under the surface $z = f(x,y)$ above the domain D.

Fubini's theorem states that the order of integration is not important if the function is continuous over the domain D. Hence:

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dxdy = \iint_D f(x,y)\,dydx$$

Example: solve the following double integral if $D = [-1,1] \times [0,1]$.

$$\begin{aligned}
\iint_D x^2 + y^2\,dxdy &= \int_0^1 \int_{-1}^1 x^2 + y^2\,dx\,dy \\
&= \int_0^1 \left[\frac{1}{3}x^3 + y^2x\right]_{-1}^1 dy \\
&= \int_0^1 \left(\frac{1}{3} + y^2\right) - \left(-\frac{1}{3} - y^2\right) dy \\
&= \int_0^1 \frac{2}{3} + 2y^2\,dy \\
&= \left[\frac{2}{3}y + \frac{2}{3}y^3\right]_0^1 \\
&= \frac{2}{3} + \frac{2}{3} \\
\iint_D x^2 + y^2\,dxdy &= \frac{4}{3}
\end{aligned}$$