Physical Cosmology Subject Notes

Key Equations

Basic Cosmology

Key constants

$$\rho_c = 136 \times 10^9 \text{ M}_{\text{solar}} \text{ Mpc}^{-3}$$
$$n_b = 2.75 \times 10^{-8} \Omega_b h^2 = 2.75 \times 10^{-8}(0.02)$$
$$T_0 = 2.73\ K$$
$$\Omega_m = 0.3, \Omega_r = 10^{-4}, \Omega_\Lambda = 0.7$$
$$h = 0.68$$
$$\frac{n_b}{n_\gamma} = 2.7 \times 10^{-8} \Omega_b h^2$$

Angles

$$1 \text{ rad} = 57.3^{\text{o}} = 3{,}438 \text{ arcmin} = 206{,}265 \text{ arcsec}$$
$$1 \text{ sphere} = 4\pi \text{ str} \approx 41{,}000 \text{ deg}^2 \approx 1.5 \times 10^8 \text{ arcmin}^2$$
$$\frac{dV}{dz} = \frac{c}{H_0} d_A^2 (1+z)^2 d\Omega = \frac{c}{H_0} d_C^2 d\Omega \text{ (in str)}$$
$$\delta \approx \frac{d_{obj}}{D_{dist}} \text{ (in rad)}$$
$$D_{tel} = \frac{1.22\lambda_{obs}}{\delta \text{ (rad)}}$$

Assorted equations

$$T(z) = T_0(1+z) = \frac{T_0}{a}$$
$$n = \frac{n_0}{(1+z)^3} = \frac{n_0}{a^3}$$
$$a = \frac{1}{1+z}$$
$$c_s = \frac{1}{\sqrt{3(1+R)}}$$
$$\lambda_{obs} = (1+z)\lambda_{em}$$
$$\frac{\rho_b}{\rho_\gamma} = \frac{\Omega_b}{\Omega_r}$$

Friedmann Equations

Key equations

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = H_0^2[\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda]$$
$$H = H_0\sqrt{\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda}$$

For $\Omega_m = 1$ we have the solution:

$$a = \left(\frac{3H_0}{2}\right)^{2/3} t^{2/3}$$

Equation of state:

$$P = w\rho$$
$$\rho \propto a^{-3(1+w)}$$

Distance Calculations

$$\frac{c}{H_0} = \frac{3 \times 10^{-8}\ \mathrm{m\,s^{-1}}}{680\ \mathrm{km\,s^{-1}\,Mpc^{-1}}} = \frac{9.716 \times 10^{-15}\ \mathrm{Mpc\,s^{-1}}}{0.22 \times 10^{-17} s^{-1}} = 4408\ \mathrm{Mpc}$$

$$r_{em} \approx 4.4\ \mathrm{Gpc} \times \int_0^{z\prime} \frac{1}{\left[\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_k(1+z)^2 + \Omega_\Lambda\right]^{1/2}} dz$$

Radiation dominated:

$$\begin{aligned}\frac{1}{\sqrt{\Omega_r}} \int_0^{z'} \frac{1}{(1+z)^2} dz &= \frac{1}{\sqrt{\Omega_r}} \int_1^{u-1} u^{-2}\, du \\ &= \frac{1}{\sqrt{\Omega_r}} \left[\frac{1}{(-1)} u^{-1}\right]_1^{u-1} \\ &= -(100)[(u-1)^{-1} - 1] \\ &= -(100)[z^{-1} - 1] \\ &= 100(1 - z^{-1})\end{aligned}$$

Matter dominated:

$$\begin{aligned}\frac{1}{\sqrt{\Omega_m}} \int_0^{z'} \frac{1}{(1+z)^{\frac{3}{2}}} dz &= \frac{1}{\sqrt{\Omega_m}} \int_1^{u-1} u^{-\frac{3}{2}} du \\ &= \frac{1}{\sqrt{\Omega_m}} \left[\frac{1}{\left(-\frac{1}{2}\right)} u^{-\frac{1}{2}}\right]_1^{u-1} \\ &= \frac{1}{\sqrt{0.3}} \left[(-2) u^{-\frac{1}{2}}\right]_1^{u-1} \\ &= -2(1.83) \left[(u-1)^{-\frac{1}{2}} - 1\right] \\ &= -3.65 \left[z^{-\frac{1}{2}} - 1\right] \\ &= 3.56\left(1 - z^{-1/2}\right)\end{aligned}$$

Age Calculations

$$t_{age} \approx 14\ \mathrm{Gy} \times \int_{z\prime}^{\infty} \frac{1}{(1+z)\left[\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_k(1+z)^2 + \Omega_\Lambda\right]^{1/2}} dz$$

Radiation dominated:

$$\frac{1}{\sqrt{\Omega_r}}\int_z^\infty \frac{1}{(1+z)^3}dz = \frac{1}{\sqrt{\Omega_r}}\int_{u-1}^\infty u^{-3}\,du$$
$$= \frac{1}{\sqrt{\Omega_r}}\left[\frac{1}{(-2)}u^{-2}\right]_{u-1}^\infty$$
$$= \frac{1}{2\sqrt{\Omega_r}}(u-1)^{-2}$$
$$= \frac{1}{2}(100)\left(\frac{1}{z^2}\right)$$
$$= 50z^{-2}$$

Matter dominated:

$$\frac{1}{\sqrt{\Omega_m}}\int_z^\infty \frac{1}{\sqrt{(1+z)^5}}dz = \frac{1}{\sqrt{\Omega_m}}\int_{u-1}^\infty u^{-5/2}\,du$$
$$= \frac{1}{\sqrt{\Omega_m}}\left[\frac{1}{\left(-\frac{3}{2}\right)}u^{-\frac{3}{2}}\right]_{u-1}^\infty$$
$$= 1.83\frac{2}{3}(u-1)^{-\frac{3}{2}}$$
$$= 1.22\left(\frac{1}{\sqrt{z}}\right)^3$$
$$= 1.22z^{-1.5}$$

Dark energy dominated:

$$\frac{1}{\sqrt{\Omega_\Lambda}}\int_z^\infty \frac{1}{1+z}dz = \frac{1}{\sqrt{\Omega_\Lambda}}\int_{u-1}^\infty \frac{1}{u}du$$
$$= \frac{1}{\sqrt{\Omega_\Lambda}}[\log(u)]_{u-1}^\infty$$
$$= \frac{1}{0.84}[\log(\infty) - \log(z)]$$

Structure Formation

Growth rate in EdS universe

$$\delta_{grow} \propto a \propto t^{\frac{2}{3}}$$

Press-Schechter mass function

$$F(>m) = 2\int_{\delta_c}^\infty \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{\delta^2}{2\sigma^2(m)}\right]d\delta = \text{erfc}\left[\frac{\delta_c}{\sqrt{2}\sigma(m)}\right]$$
$$dn = -\sqrt{\frac{2}{\pi}}\frac{\rho_c\Omega_m}{m}\frac{d\sigma}{dm}\frac{\delta_c}{\sigma^2}\exp\left[-\frac{\delta_c^2}{2\sigma^2}\right] \times dm$$
$$\sigma(M) = M^{-(n+3)/6}, \sigma_8 = 0.815$$

$$dn = \sqrt{\frac{2}{\pi}}\frac{\rho_m}{m^2}\left|\frac{n+3}{6}\right|\frac{\delta_c}{\sigma}\exp\left[-\frac{1}{2}\left(\frac{\delta_c}{\sigma}\right)^2\right] \times dm$$

Mass to radius

$$M = \frac{4\pi}{3}\rho_c \Omega_m R^3$$

Baryonic density growth equations

$$\frac{\partial^2 \delta_k^{dm}}{\delta t^2} + 2H(t)\frac{\partial \delta_k^{dm}}{\partial t} = 4\pi G\left(\rho_{dm}\delta_k^{dm} + \rho_b \delta_k^b\right) - \frac{k^2 C_s^2}{a^2}\delta_k^b$$
$$\frac{\partial^2 \delta_k^{b}}{\delta t^2} + 2H(t)\frac{\partial \delta_k^{b}}{\partial t} = 4\pi G\left(\rho_{dm}\delta_k^{dm} + \rho_b \delta_k^b\right)$$

Virialisation temperature

$$T_{vir} = 2.3 \times 10^5 (1+z)\left(\frac{M}{10^{12} M_{solar}}\right)^{2/3} h_{50}^{2/3}\, K$$

The surface of last scattering (sound horizon)

$$r_{sls} = \frac{1}{2}\int_{z_{LS}}^{\infty} \frac{1}{H} dz$$

Thermodynamics

Particle masses

$$m_p = 938.3 \text{ MeV}$$
$$m_n = 939.6 \text{ MeV}$$
$$m_e = 0.511 \text{ MeV}$$

Freezout temperatures

$$T_{dec}(\gamma \leftrightarrow WIMP) = 100 \text{ GeV}$$
$$T_{dec}(2\gamma \leftrightarrow p + \bar{p}) = 22 \text{ MeV}$$
$$T_{dec}(n + \nu_e \leftrightarrow p + e) = 0.8 \text{ MeV}$$
$$T_{dec}(p + e \rightarrow H + \gamma) = 1 \text{ eV}$$

High temperatures (relativistic)

Bosons

$$\rho = \left(\frac{\pi^2}{30}\right) g T^4$$
$$n = \left(\frac{1.2}{\pi^2}\right) g T^3$$
$$P = \frac{\rho}{3}$$

Fermions

$$\rho = \frac{7}{8}\left(\frac{\pi^2}{30}\right) g T^4$$
$$n = \frac{3}{4}\left(\frac{1.2}{\pi^2}\right) g T^3$$
$$P = \frac{\rho}{3}$$

Low temperatures (non-relativistic)

$$\rho = nm$$
$$n = g\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{(m-\mu)}{T}}$$
$$P = nT$$

Saha equation

$$\frac{1 - x_e}{x_e^2} = \eta \left(4\zeta(3) \sqrt{\frac{2}{\pi}} \right) \left(\frac{T}{m_e} \right)^{3/2} e^{-B/T}$$

Basic Cosmology

History of the Universe

- 10^{-35} s: inflation (10^{16} GeV)
- 10^{-8} s: WIMP freezout fixing amount of dark matter (100 GeV)
- 0.1 s: neutrino decoupling from photon-baryon gas (3 MeV)
- 1 min: Big Bang nucleosynthesis (1 MeV)
- 10,000 years: matter-radiation equality, beginning of non-relativistic domination (3 eV)
- 400,000 years: recombination/CMB, decoupling of photons from baryons (1 eV)
- 4 million years: decoupling of baryons from photons, start of baryon structure growth (0.1 eV)
- 500 million years: first stars and galaxies (0.001 eV)
- 1 billion years: reionisation (0.001 eV)
- 14 billion years: today (0.0001 eV)

The thermal history of the universe

Event	T	redshift	time	size of today's universe
Now	2.73 K	0	13 Gyr	13×10^9 light-years
Distant galaxy	16 K	5	1 Gyr	7×10^9 light-years
Recombination	3000 K	1100	$10^{5.6}$ years	11×10^6 light-years
Radiation domination	9500 K	3500	$10^{4.7}$ years	4×10^6 light-years
Electron pair threshold	$10^{9.7}$ K	$10^{9.5}$	3 s	4 light-years
Nucleosynthesis	10^{10} K	10^{10}	1 s	1.3 light-years
Nucleon pair threshold	10^{13} K	10^{13}	$10^{-6.6}$ s	0.5 light-day
Electroweak unification	$10^{15.5}$ K	10^{15}	10^{-12} s	0.1 light-hour
Grand unification	10^{28} K	10^{28}	10^{-36} s	10^{-2} m
Quantum gravity	10^{32} K	10^{32}	10^{-43} s	10^{-6} m

Deriving Hubble's Law

If we begin with the only large-scale motion relationship consistent with both isotropy and homogeneity:

$$\dot{r} = f(t) r$$

Where we have $r = a(t)x$, we can write:

$$\begin{aligned} r &= \frac{\dot{r}}{f(t)} \\ &= \frac{\dot{(ax)}}{f(t)} \end{aligned}$$

$$= \frac{\dot{a}x + a\dot{x}}{f(t)}$$

Since x is the commoving coordinate it does not change (on average) over time, meaning that $\dot{x} = 0$ and so we have:

$$r = \frac{\dot{a}x}{f(t)}$$
$$f(t) = \frac{\dot{a}x}{r}$$
$$= \frac{\dot{a}x}{ax}$$
$$f(t) = \frac{\dot{a}}{a}$$

This gives us Hubble's law:

$$\dot{r} = \frac{\dot{a}}{a}r = H(t)r$$

Friedman Equations

Apply for Homogenous, Isotropic Universe

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$
$$\left(\frac{\dot{a}}{a}\right)^2 = -8\pi G P - \frac{k}{a^2} - \frac{2\ddot{a}}{a}$$

To solve these we need an equation of state specifying $P = f(\rho)$, and also to specify the geometry factor k.

The Expansion Rate

A faster rate of expansion increases the rate at which structures are pulled apart, thereby slowing structure growth. A faster expansion rate also means that the universe was relatively smaller in the past compared to a slower expansion rate, meaning we would see *fewer* structures per volume element due to changes in the volume element. However faster expansion in the past also means that there will have been slower structure growth, implying a *higher* number density in the past (it grew quicker to match today's value). These two effects thus operate in opposite directions.

Equations of State

We always assume linear equations of state taking the form:

$$P = w\rho$$

The value of w depends upon the type of energy in question:

- Non-relativistic matter (baryons and CDM): $w = 0$
- Radiation (photons or hot dark matter): $w = \frac{1}{3}$
- Cosmological constant Λ: $w = -1$

Pressure Evolution

Using the first of Friedmann's equations and the equation of state given above:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$
$$\frac{8\pi G}{3}\rho a^3 = a(\dot{a}^2 + k)$$
$$\frac{d}{dt}(\rho a^3) = \frac{3}{8\pi G}\frac{d}{dt}\left(a(\dot{a}^2 + k)\right)$$
$$= \frac{3}{8\pi G}[\dot{a}(\dot{a}^2 + k) + a(2\ddot{a}\dot{a})]$$
$$\frac{d}{dt}(\rho a^3) = \frac{3\dot{a}a^2}{8\pi G}\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} + \frac{2\ddot{a}}{a}\right]$$

Substituting in Friedmann's second equation we have:

$$\frac{d}{dt}(\rho a^3) = \frac{3\dot{a}a^2}{8\pi G}(-8\pi GP)$$
$$\frac{d}{da}\frac{da}{dt}(\rho a^3) = -3\dot{a}a^2 P$$
$$\frac{d}{da}\dot{a}(\rho a^3) = -3\dot{a}a^2 P$$
$$\frac{d}{da}(\rho a^3) = -3w\rho a^2$$
$$\frac{d\rho}{da}a^3 + 3\rho a^2 = -3w\rho a^2$$
$$\frac{d\rho}{da}a^3 = -3a^2(1+w)\rho$$
$$\frac{d\rho}{da} = -3(1+w)\frac{\rho}{a}$$
$$\frac{d\rho}{\rho} = -3(1+w)\frac{da}{a}$$
$$\log\rho = -3(1+w)\log a + A$$
$$\rho = Aa^{-3(1+w)}$$

Critical Density

The critical density of the universe is the energy density that results in a flat universe, such that $k = 0$. We can write it in terms of its components:

$$\frac{\rho}{\rho_c} = \frac{\rho_m}{\rho_c} + \frac{\rho_r}{\rho_c} + \frac{\rho_\Lambda}{\rho_c}$$
$$\Omega = \Omega_m + \Omega_r + \Omega_\Lambda$$

We can express this in terms of present-day values of these densities:

$$\Omega = \Omega_{m,0}\left(\frac{a_0}{a}\right)^3 + \Omega_{r,0}\left(\frac{a_0}{a}\right)^4 + \Omega_\Lambda$$
$$\rho(t) = \rho_{c,0}\left[\Omega_{m,0}\left(\frac{a_0}{a}\right)^3 + \Omega_{r,0}\left(\frac{a_0}{a}\right)^4 + \Omega_\Lambda\right]$$

Using Friedmann's first equation we have:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho(t)$$
$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho_{c,0}\left[\Omega_{m,0}\left(\frac{a_0}{a}\right)^3 + \Omega_{r,0}\left(\frac{a_0}{a}\right)^4 + \Omega_\Lambda\right]$$

Using the same equation again we know that:

$$\frac{8\pi G}{3}\rho_{c,0} = \left(\frac{\dot{a}_0}{a_0}\right)^2 = H_0^2$$

Thus we arrive at:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = H_0^2\left[\Omega_{m,0}\left(\frac{a_0}{a}\right)^3 + \Omega_{r,0}\left(\frac{a_0}{a}\right)^4 + \Omega_\Lambda\right]$$

For a flat universe and using the convention $a_0 = 1$ we have the simplification:

$$H^2 = H_0^2\left[\Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4} + \Omega_\Lambda\right]$$

Solving for the expansion rate

For a flat model with $\Omega_m = 1$ the solution is simple:

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 a^{-3}$$
$$\dot{a}^2 = \frac{H_0^2}{a}$$
$$\dot{a} = \frac{H_0}{a^{\frac{1}{2}}}$$
$$\frac{dt}{da} = \frac{1}{H_0}a^{\frac{1}{2}}$$
$$t = \frac{2}{3}\frac{1}{H_0}a^{\frac{3}{2}} + C$$
$$a = \left(\frac{3H_0}{2}\right)^{2/3} t^{2/3}$$

Redshift

Cosmological redshift is caused by the expansion of the universe, and is defined as:

$$1 + z(t) = \frac{a_0}{a}$$

For the usual convention $a_0 = 1$ this becomes:

$$a = \frac{1}{1+z}$$

We can thus write:

$$H = H_0\left[\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_\Lambda\right]^{1/2}$$

Distance and Time

These use the relation:

$$\frac{da}{dz} = -\frac{1}{(1+z)^2}$$

As well as the flat-universe metric:

$$c^2 dt^2 = a^2(t) dr^2$$

Distance at redshift z:

$$\begin{aligned} dr &= \frac{dr}{da}\frac{da}{dz}dz \\ &= -\frac{cdt/a}{da}\frac{1}{(1+z)^2}dz \\ &= -\frac{c}{a}\frac{dt}{da}\frac{1}{(1+z)^2}dz \\ &= -c(1+z)\frac{1}{\dot{a}}\frac{1}{(1+z)^2}dz \\ &= -c\frac{a}{\dot{a}}dz \\ dr &= -\frac{c}{H}dz \end{aligned}$$

$$\begin{aligned} r_{em} &= \int_0^{a(z)} dr \\ &= -\int_z^0 \frac{c}{H}dz \\ &= \frac{c}{H_0}\int_0^{z'} \frac{1}{\left[\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_\Lambda\right]^{1/2}}dz \\ r_{em} &\approx 4.4\ \text{Gpc} \times \int_0^{z'} \frac{1}{\left[\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_\Lambda\right]^{1/2}}dz \end{aligned}$$

Age at redshift z:

$$\begin{aligned} dt &= \frac{dt}{da}\frac{da}{dz}\ dz \\ &= -\frac{1}{\dot{a}}\frac{1}{(1+z)^2}\ dz \\ dt &= -\frac{1}{H}\frac{1}{1+z}dz \end{aligned}$$

$$\begin{aligned} t_{age} &= \int_0^{a(z)} dt \\ &= \int_0^{a(z)} -\frac{1}{H}\frac{1}{1+z}dz \\ &= \frac{1}{H_0}\int_{z'}^{\infty} \frac{1}{(1+z)\left[\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_\Lambda\right]^{1/2}}dz \\ t_{age} &\approx 14\ \text{Gy} \times \int_{z'}^{\infty} \frac{1}{(1+z)\left[\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_\Lambda\right]^{1/2}}dz \end{aligned}$$

Distance Measures

Proper distance

Analogous to ruler distance, with measurements made along null-geodesics.

$$D_p = \frac{D_C}{1+z}$$

Comoving distance

The distance between two objects moving with the Hubble flow. It will not change over time as a result of the expansion of the universe. It is the proper distance multiplied by $(1+z)$.

$$D_C = \int_0^z \frac{c}{H(z)} dz = \frac{c}{H_0} \int_0^z \frac{1}{\sqrt{\Omega_m (1+z)^3 + \Omega_k (1+z)^2 + \Omega_\Lambda}} dz'$$

Angular diameter distance

The angular diameter distance is the ratio of an object's physical transverse size to its angular size (in radians). It depends on the universe geometry:

$$D_A = \frac{d_{obj}}{\delta} = \frac{1}{1+z} \begin{cases} \frac{c}{H_0} \frac{1}{\sqrt{\Omega_k}} \sinh\left(\sqrt{\Omega_k} \frac{D_C}{D_H}\right), & \Omega_k > 0 \\ D_C, & \Omega_k = 0 \\ \frac{c}{H_0} \frac{1}{\sqrt{\Omega_k}} \sin\left(\sqrt{\Omega_k} \frac{D_C}{D_H}\right), & \Omega_k < 0 \end{cases}$$

It is famous for not increasing indefinitely; it turns over at z ~ 1 and thereafter more distant objects actually appear larger.

Luminosity distance

The luminosity distance is the ratio between the emitted luminosity of a source L_{em} and the received flux by the viewer F_{rec}.Received luminosity is related to emitted luminosity by $L_{Rec} = L_{em}/(1+z)^2$, with one power due to time dilation (photons arriving slower) and one due to redshift (lower energy).

This allows us to relate it to other distance measures:

$$D_L = \sqrt{\frac{L}{4\pi}} = (1+z) D_C = (1+z)^2 D_A$$

Inflation

Problems with hot big bang Model

Small-scale horizon problem

The cosmic horizon (causally connected region) is given by:

$$d_H = \frac{c}{H} = \frac{c}{H_0\left(\Omega_{m,0} a^{-3} + \Omega_{r,0} a^{-4}\right)}$$

This means that d_H grows proportionally between $a^{1.5} < d_H < a^2$. However, structures (or density perturbations that are the precursors of structures) only grow proportional to the scale factor ($\lambda \propto a$). This means that at some point shortly after the big bang, there was a cross-over point, before which the characteristic length of primordial fluctuations λ would have been smaller than the cosmic horizon d_H. This is impossible because no causally-connected structure can be larger than the cosmic horizon.

Large-scale horizon problem

The characteristic fluctuation size entering the horizon at any given time is given by:

$$\lambda = d_H(a)$$
$$\lambda = \frac{c}{H_0\left(\Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4}\right)}$$

The universe is homogenous on large scales, including regions that have never been in causal contact. How is this possible?

Flatness problem

If $\Omega \approx 1$ at present, it must have been exceptionally close to 1 in the past. Why should this be the case? From:

$$\frac{k}{a^2H^2} = \Omega(t) - 1$$

We can derive:

$$\Omega(t) - 1 = \frac{\Omega_0 - 1}{\Omega_m a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda^2 + (1 - \Omega)}$$

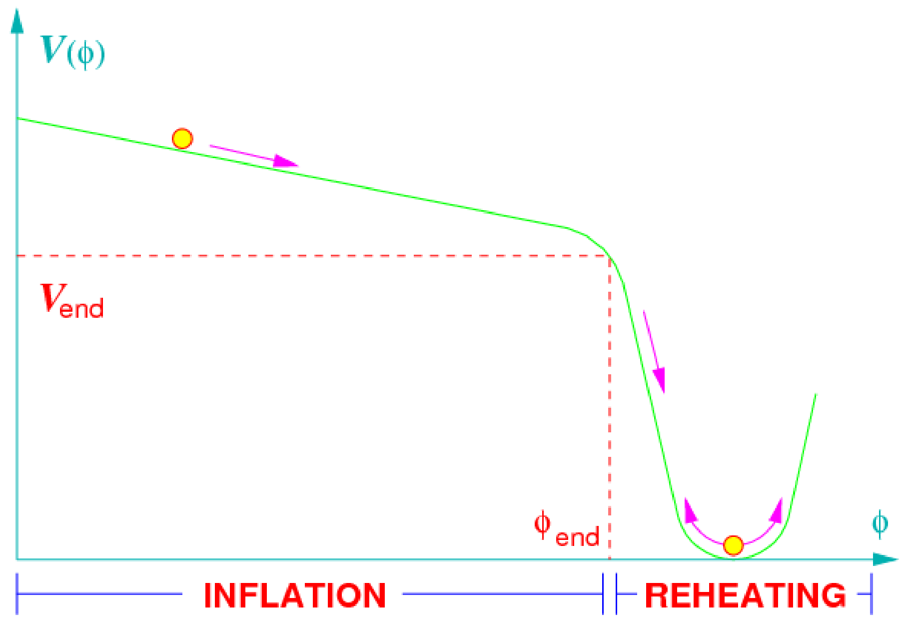
In the limit $a \to 0$, the RHS approaches zero, meaning $\Omega(0) \to 1$ regardless of what Ω_0. This means that flatness only ever decreases over time, so if the universe is nearly flat now, it must have been extremely flat in the past.

The Physics of Inflation

In order for inflation to occur, the energy density of the universe must be dominated by a negative pressure component such that $\ddot{a} > 0$, for example the cosmological constant. This is in fact the definition of inflation: $\frac{d^2a}{dt^2} > 0$. This is usually thought to be cause by a scalar field called the Inflaton Field. The pressure of such a field with energy density ϕ is given by:

$$\rho = KE + PE = \frac{1}{2}\left(\dot{\phi}\right)^2 + V(\phi)$$
$$P = KE - PE = \frac{1}{2}\left(\dot{\phi}\right)^2 - V(\phi)$$

If the potential energy term is greater than the kinetic term then the pressure is negative, which is the result we want. In the slow roll model of inflation, the mostly flat portion of the potential energy curve eventually gives out to a deep well, resulting in reheating of the universe.



The slow-roll conditions necessary for this type of inflation to occur are given by:

$$\epsilon(\phi) = \frac{1}{16\pi G}\left(\frac{1}{V}\frac{dV}{d\phi}\right)^2 \ll 1$$
$$|\eta(\phi)| = \frac{1}{8\pi G}\left(\frac{1}{V}\frac{d^2V}{d\phi^2}\right) \ll 1$$

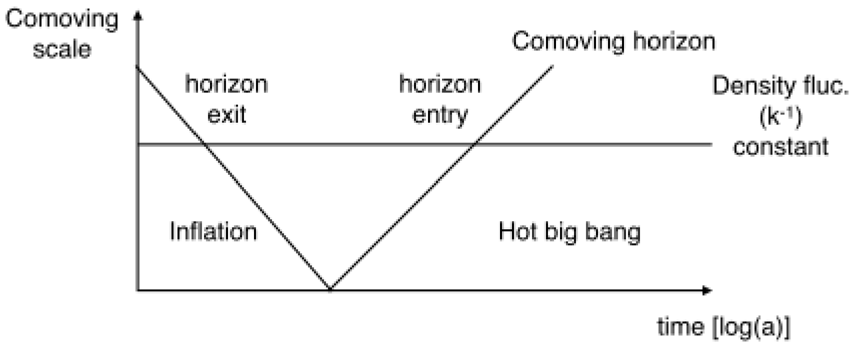
This means that the power distribution of primordial density fluctuations will be given by:

$$k^{n_s-1}, n_s = 1 - 6\epsilon - 2\eta$$

For slow roll inflation, this corresponds to a nearly scale-invariant distribution. However a perfectly constant inflaton potential is equivalent to a cosmological constant, and leads to never-ending exponential expansion. Our observational constraint that inflation ended means the primordial power spectrum should not be exactly scale-invariant.

How inflation solves problems

- Flatness problem: inflation increases the scale factor by some e^{60}, which (just like a big enough sphere) looks flat locally.
- Small-scale horizon problem: during inflation λ grew much more quickly than a, so before inflation the entire fluctuation region would have been inside the horizon.
- Large-scale horizon problem: although distant parts of the universe are causally disconnected today, before inflation they were compressed into a very small space, allowing them to interact.
- Missing relics: magnetic monopoles are predicted by many GUT theories, however none have been observed. Inflation resolves this because such particles are predicted to be formed at pre-inflation energy scales, and so during inflation their density will be diluted to almost zero.



Conditions for Inflation

Beginning with Friedman's first equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$$
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + V(\phi)\right)$$

The continuity equation is given by:

$$a\frac{d\rho}{da} = -3(\rho + P)$$

Substituting in the forms of ρ and P we have:

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi}$$

Now the condition for inflation to occur is $\ddot{a} > 0$. This is equivalent to:

$$\rho + 3P < 0$$

Substituting in for the values of ρ and P:

$$\frac{1}{2}(\dot{\phi})^2 + V(\phi) + \frac{3}{2}(\dot{\phi})^2 - 3V(\phi) < 0$$
$$\frac{1}{2}(\dot{\phi})^2 < V(\phi)$$

Which reproduces the result stated above: the kinetic term has to be less than the potential term for inflation to occur.

Slow Roll Inflation

Inflation does not necessarily imply exponential growth of the scale factor, but slow roll inflation does. We can show this by noting slow role implies V is effectively constant, and thus the first Friedmann equation becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}V$$

$$\frac{da}{dt} = a\sqrt{\frac{8\pi G}{3}V}$$

$$\frac{1}{a}da = \sqrt{\frac{8\pi G}{3}V}\,dt$$

$$\log a = \sqrt{\frac{8\pi G}{3}V}\,t$$

$$a = \exp\left(\sqrt{\frac{8\pi G}{3}V}\,t\right)$$

This also implies that H is constant during inflation.

Reheating

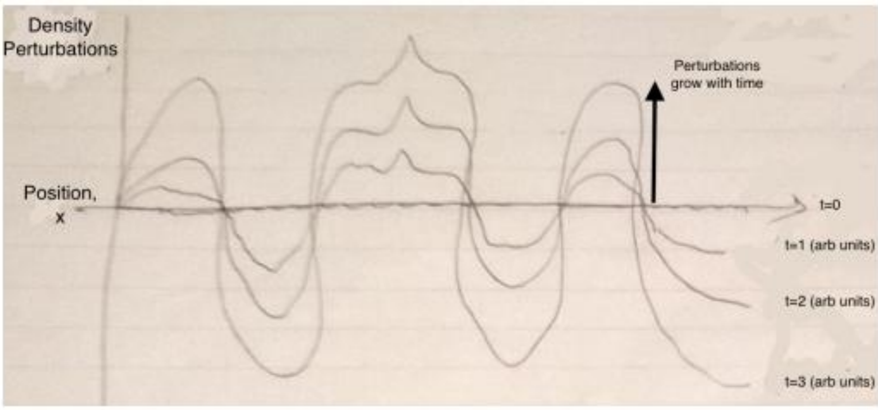
Reheating is the process whereby the inflaton's energy density is converted back into conventional matter after inflation, re-entering the standard big bang theory.

Dark Matter Perturbations

Linear Structure Growth

Large-scale structure (LSS) typically refers to the largest scales where the growth is linear or just beginning to be non-linear growth. On smaller scales, we have collapsed objects: the largest of these are galaxy clusters.

In ΛCDM, gravity drives structure growth: over-dense regions exert a gravitational pull on their surroundings. Said surroundings fall into the over-dense region, making it even denser. It then exerts an even stronger gravitational pull, accelerating the process by which it accretes mass. Expansion pushes the opposite direction, but otherwise leaves the picture unchanged. In any delta time, mass falls inwards a bit due to gravity and is pulled apart a bit due to the expansion. With fast enough expansion, you get a big rip scenario where no structures form.



Here we will assume that density perturbations are small enough for a linear analysis to be used, and that the universe can be described by a pressureless fluid (so after recombination). Under these conditions density scales as:

$$\rho \propto a^{-3}$$

We define a density contrast and scale factor contrasts as:

$$\delta = \frac{\rho - \rho_b}{\rho_b}$$
$$\epsilon = \frac{a - a_b}{a_b}$$

Since conservation of mass gives $\rho a^3 = C$ we can determine that $\delta = 3\epsilon$. Considering these definitions and given the Friedmann equations, we find that there are two time-varying paths of the mass perturbations, one growing and one decaying.

$$\delta_{grow} = \frac{3\dot{a}\Delta k}{2a_b} \int_0^{a_b} (\dot{a}^{-3})\, da$$
$$\delta_{dec} = \frac{3\dot{a}\Delta t_c}{a_b}$$

Where t_c is the difference between the age of the over-density and the age of the background universe, and k is the curvature factor. In an Einstein-de Sitter universe, we have $\dot{a}_b^2 \propto 1/a_b$.

$$\delta_{grow} \propto \frac{\Delta k}{a_b^{3/2}} \int_0^{a_b} a^{3/2}\, da \propto a_b \propto t^{\frac{2}{3}}$$
$$\delta_{dec} \propto \frac{1}{a_b^{3/2}} \propto \frac{1}{t}$$

So in the linear growth mode, perpetrations grow linearly with the scale factor.

We define an object formation time as the point at which the boundary begins to collapse, ie when $\dot{R} = 0$. This happens at the turnaround time, when it reaches its maximum radius, tmax.

Non-Linear Growth

We can identify two important epochs in an objects lifetime: its formation time (when it leaves the Hubble flow, $\dot{a} = 0$) and its virialization time (identified with the collapse time when nominally R=0 in the simple spherical model).

If we have an isolated overdense region of the universe, we can represent the equation for the locl 'pocket universe' as:

$$H_{over}^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2}$$

Where $k = 1$ for the closed mini-universe. The universe outside will have $\rho = \rho_b$ and $k = 0$, and thus we find that:

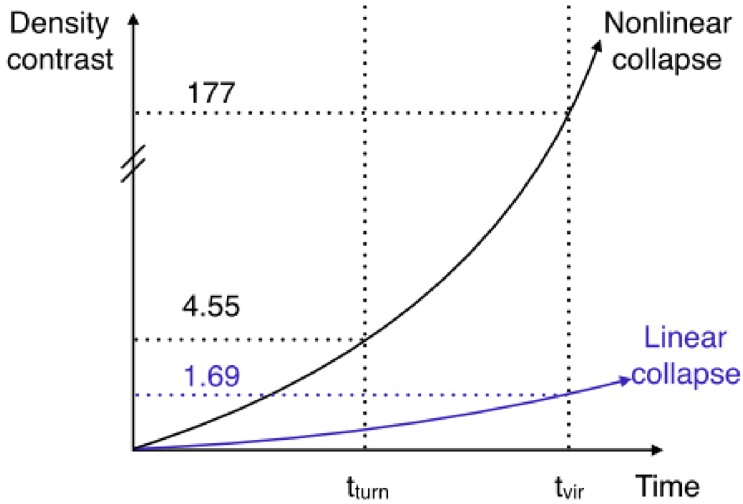
$$H_{over} < H$$

This means that the mini-universe will slow down with respect to the hubble flow and eventually reverse to a contraction. The reason this doesn't result in a universe full of black holes is because small perturbations in our sphere will be magnified during collapse, violating symmetry and thus preventing complete collapse. Instead, the potential energy is converted into kinetic energy to lead to a stable halo, a process called virialisation. A stable halo, self-supported by its kinetic energy is a virialized halo.

A virialised halo has the following properties in an EdS universe:

$$R_{vir} = 0.5R_{max}$$
$$t_{vir} = 2t_{turn}$$
$$\rho_{vir} = 8\rho_{turn}$$
$$\delta_{turn} = \frac{\rho_{turn}}{\rho_b(t_{turn})} = 4.55$$
$$\delta_{col} = \frac{\rho}{\rho_b} = 18\pi^2 \approx 178$$

If instead of collapsing and virialising the overdensity had stayed linear (i.e. $\delta \propto a$ in EdS), it would have grown to 1.69.



Press-Schechter Mass Function

The Press-Schecter model assumes that initial density perturbations are a Gaussian random field in which every frequency mode is independent, with a nearly scale invariant power spectrum. It also assumes spherical halos. These initial modes have the following properties:

$$\delta_k = \int x\delta(\bar{x})e^{ik\bar{x}}\, d^3x$$
$$k = \frac{2\pi}{\lambda}$$
$$P(k) = |\delta_k|^2 = \frac{1}{k^3}k^{n_s-1}$$

Where $n_s = 0.96$, essentially the flat spectrum predicted by inflation. The power spectrum at a later redshift z is given by:

$$P(k,z) = P_0(k)T^2(k,z)$$

The Press-Schechter theory postulates that the fraction of mass in collapsed objects with mass larger than $m = \frac{4}{3}\rho\pi R^3$ is equal to the fraction of dark matter that is in overdensities of $\delta > \delta_c$, where δ_c depends upon redshift. We can write this as (note the factor of two to remove negative part of the probability distribution):

$$F(> m) = 2\int_{\delta_c}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\delta^2}{2\sigma^2(m)}\right] d\delta = \mathrm{erfc}\left[\frac{\delta_c}{\sqrt{2}\sigma(m)}\right]$$

The variance $\sigma(m)$ depends upon how we choose to smooth the underlying density fluctuation distribution. We smooth it using a smoothing kernel:

$$W(r) = \frac{3}{4\pi R^3}, r \leq R;\ 0 \text{ otherwise}$$

This means that fluctuations less than a certain size, or equivalently corresponding to halos of a certain mass, will not be apparent – only larger fluctuations will show up. Thus the variance of the distribution changes depending on the mass scaling we use. The higher the mass we pick, the more the distribution is smoothed, and so the smaller the variance will be. It scales with mass as:

$$\sigma(M) = M^{-(n+3)/6}$$

$$n_{eff} \approx -1.5 \text{ for clusters}, -2 \text{ for galaxies}$$

Note that m doesn't change δ which always has a mean of zero and a critical value (at the present day) of 1.69. The result is affected by m only via its effect on $\sigma(m)$, which gets smaller as m increases, thereby representing the fact that large fluctuations are less likely as we smoother the density fluctuation δ over a wider space (corresponding to a bigger mass). A commonly used mass smoothing is a length scale of $R = 8h^{-1}$ Mpc (cluster size), the corresponding variance is $\sigma_8 \approx 0.8$. We scale this:

$$\sigma = \sigma_8 \left(\frac{M}{M_8 = 2.6 \times 10^{14} h^{-1}\ M_{solar}} \right)^{-(n+3)/6}$$

$$M = \frac{4}{3} \rho_c \Omega_m \pi R^3$$

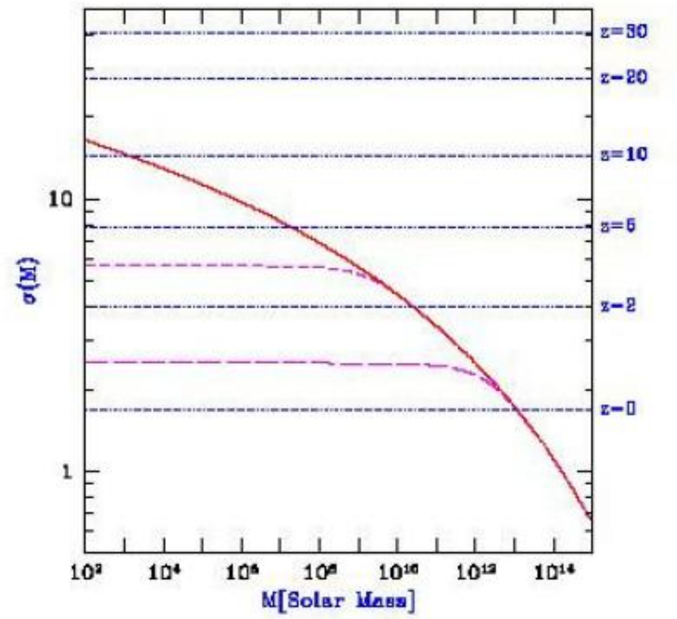
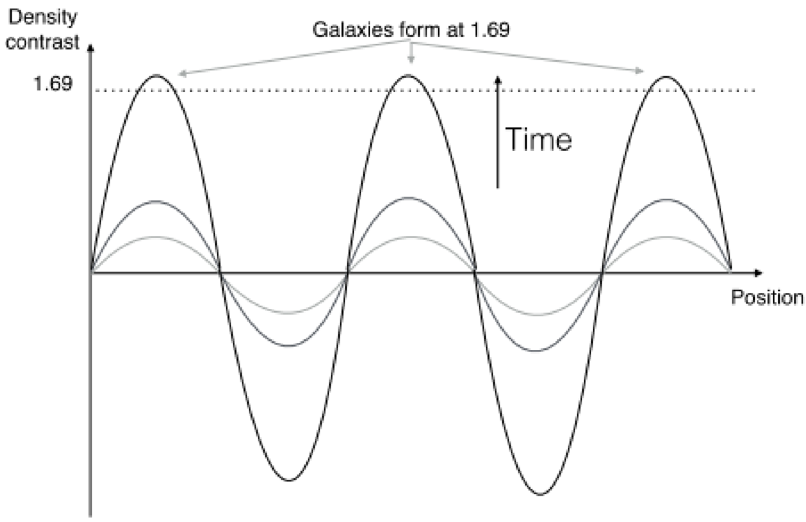


Fig. 5.— Mass fluctuations and collapse thresholds in cold dark matter models. The horizontal dotted lines show the value of the extrapolated collapse overdensity $\delta_{crit}(z)$ at the indicated redshifts. Also shown is the value of $\sigma(M)$ for the cosmological parameters given in the text (solid curve), as well as $\sigma(M)$ for a power spectrum with a cutoff below a mass $M = 1.7 \times 10^8 M_\odot$ (short-dashed curve), or $M = 1.7 \times 10^{11} M_\odot$ (long-dashed curve). The intersection of the horizontal lines with the other curves indicate, at each redshift z, the mass scale (for each model) at which a $1-\sigma$ fluctuation is just collapsing at z (see the discussion in the text).



To find the number density of objects in the mass range m to $m + dm$ we use:

$$
\begin{aligned}
dn &= \frac{2\rho}{m}[F(>m) - F(>m+dm)] \\
&= -\frac{2\rho}{m}\left[\frac{\partial F(>m)}{\partial m}dm\right] \\
&= -\frac{2\rho}{m}\left[\frac{\partial F(>m)}{\partial \sigma}\frac{\partial \sigma}{\partial m}\right]dm \\
&= -2\rho\left(\frac{\delta}{\sigma^2}\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{\delta^2}{2\sigma^2}\right]\right)\frac{1}{m}dm\frac{d\sigma}{dm}
\end{aligned}
$$

$$dn = -\sqrt{\frac{2}{\pi}\frac{\rho_M}{m}\frac{\delta_c}{\sigma^2}\frac{d\sigma}{dm}}\exp\left[-\frac{\delta_c^2}{2\sigma^2}\right] \times dm$$

$$\frac{dn}{d\log(m)} = -\sqrt{\frac{2}{\pi}}\rho_M\frac{\delta_c}{\sigma^2}\exp\left[-\frac{\delta_c^2}{2\sigma^2}\right]d\sigma$$

$$\frac{N}{V} = \frac{dn}{d\log m} = \sqrt{\frac{2}{\pi}}\frac{\rho_m}{m}\left|\frac{n+3}{6}\right|\frac{\delta_c}{\sigma}\exp\left[-\frac{\delta_c^2}{2\sigma^2}\right]$$

Where n is the number density, and $\rho_M = \rho_c\Omega_m = \frac{m}{V}$ for the halo. Note that this equation will only be valid for a particular redshift, since ρ_M and δ_c will depend on the redshift per the equation:

$$\delta_c(z) = 1.69(1+z)$$

Yielding number ratios at various redshifts as:

$$\frac{N_z}{N_{z_0}} = \frac{\operatorname{erfc}\left[\frac{1.69(1+z)}{\sqrt{2}\sigma(m)}\right]}{\operatorname{erfc}\left[\frac{1.69}{\sqrt{2}\sigma(m)}\right]}$$

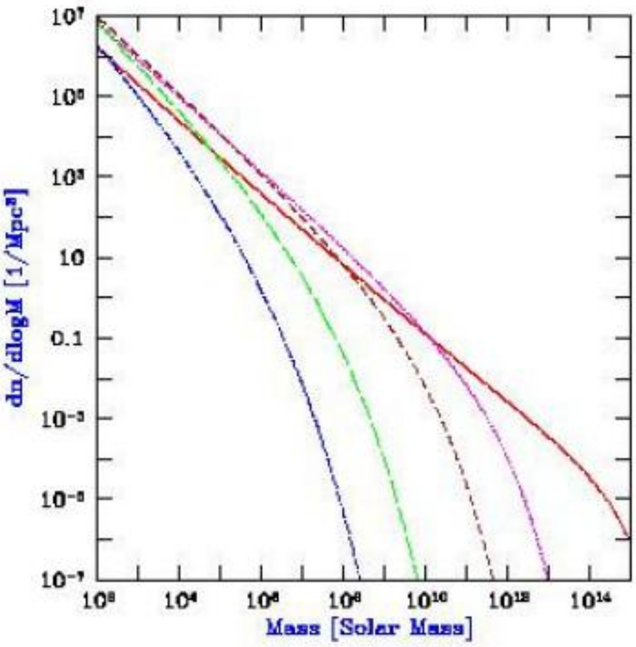


Fig. 10.— Halo mass function at several redshifts: $z = 0$ (solid curve), $z = 5$ (dotted curve), $z = 10$ (short-dashed curve), $z = 20$ (long-dashed curve), and $z = 30$ (dot-dashed curve).

Some alternate forms of Press-Schechter:

$$dn\ mdm = -2\rho \frac{\delta}{\sigma}\frac{1}{\sqrt{2\pi}} \exp\left[\frac{\delta^2}{2\sigma^2}\right] \frac{d\sigma}{\sigma}$$

$$dn\ mdm = -2\rho \frac{\delta}{\sigma}\frac{1}{\sqrt{2\pi}} \exp\left[\frac{\delta^2}{2\sigma^2}\right] \frac{d\ln(\sigma)}{d\sigma} d\sigma$$

$$dn\ = -2\rho \frac{\delta}{\sigma}\frac{1}{\sqrt{2\pi}} \exp\left[\frac{\delta^2}{2\sigma^2}\right] \frac{d\ln(\sigma)}{mdm}$$

$$dn\ = -2\rho \frac{\delta}{\sigma}\frac{1}{\sqrt{2\pi}} \exp\left[\frac{\delta^2}{2\sigma^2}\right] \frac{1}{m^2} d\ln(\sigma) \frac{m}{dm}$$

$$dn\ = -2\rho \frac{\delta}{\sigma}\frac{1}{\sqrt{2\pi}} \exp\left[\frac{\delta^2}{2\sigma^2}\right] \frac{1}{m^2} d\ln(\sigma) \left(\frac{1}{d\ln(m)}\right)$$

$$dn\ = -2\rho \frac{\delta}{\sigma}\frac{1}{\sqrt{2\pi}} \exp\left[\frac{\delta^2}{2\sigma^2}\right] \frac{1}{m^2} \frac{d\ln(\sigma)}{d\ln(m)}$$

$$dn = -\sqrt{\frac{2}{\pi}\frac{\rho_m}{m}\frac{1}{\sigma}\frac{d\sigma}{dm}\frac{\delta_c}{\sigma}} \exp\left[-\frac{\delta_c^2}{2\sigma^2}\right] \times dm$$

$$dn = -\sqrt{\frac{2}{\pi}\frac{\rho_m}{m}\frac{1}{\sigma}} \left(-\frac{n+3}{6}\right) m^{-(n+3)/6} m^{-1} \frac{\delta_c}{\sigma} \exp\left[-\frac{\delta_c^2}{2\sigma^2}\right] \times dm$$

$$dn = \sqrt{\frac{2}{\pi}\frac{\rho_m}{m}\frac{1}{\sigma}} \left|\frac{n+3}{6}\right| \sigma \frac{\delta_c}{\sigma} \exp\left[-\frac{\delta_c^2}{2\sigma^2}\right] \times \frac{dm}{m}$$

Observational Constraints

Higher mass objects like galaxy clusters are often more useful for constraining properties like dark matter content, for a number of reasons:

- More recent formation during a time when dark energy is more dominant
- Large clusters are more dominated than gravitational binding energy compared to galaxies
- Number counts are very sensitive to cosmological parameters at the high end of masses – very big difference between 10 and 11 sigma event compared to 0.5 and 0.55

Typically what is actually measured is the variable:

$$\frac{dN}{dMdzd\Omega}$$

Which refers to counts of objects of a certain mass per area in the sky per redshift.

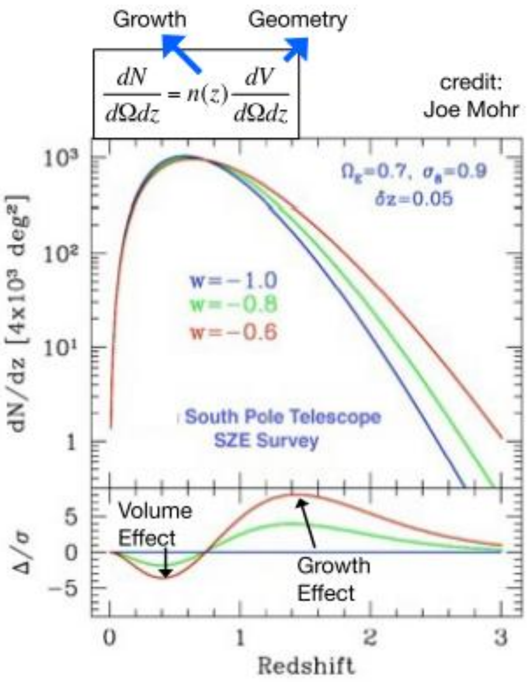


Fig. 11.—: A prediction for the abundance of galaxy clusters as the dark energy equation of state w is varied. Increasing w means the acceleration due to DE is lower, so the Universe was expanding more rapidly in the past. This means that growth is slower (so clusters that exist today formed earlier) and that the volume at a given redshift is smaller. Image credit: Joe Mohr

Neutrinos and Perturbations Example

Assuming that neutrino gravitational effects are minimal, and that we are working above the baryonic Jeans wavelength, but below the neutrino Jeans wavelength, we have the equation:

$$\left(\frac{\partial^2}{\partial t^2} + \frac{2\dot{a}}{a}\frac{\partial}{\partial t}\right)\delta_k^{bdm} = 4\pi G\rho\Omega_{bdm}\delta_k^{bdm}$$

Using the results $\frac{2\dot{a}}{a} = 4/3t$ and $4\pi G\rho = 2/3t^2$ for an Einstein-de-Sitter universe, we have:

$$\left(\frac{\partial^2}{\partial t^2} + \frac{4}{3t}\frac{\partial}{\partial t}\right)\delta_k^{bdm} = \frac{2}{3t^2}\Omega_{bdm}\delta_k^{bdm}$$

Try solution of the form $\delta_k^{bdm} = Xt^\gamma$:

$$\left(\frac{\partial^2}{\partial t^2} + \frac{4}{3t}\frac{\partial}{\partial t}\right)Xt^\gamma = \frac{2\Omega_{bdm}}{3t^2}Xt^\gamma$$

$$\gamma(\gamma-1)Xt^{\gamma-2}+\frac{4}{3t}X\gamma t^{\gamma-1}=\frac{2\Omega_{bdm}}{3t^2}Xt^{\gamma}$$
$$\gamma(\gamma-1)t^{\gamma-2}+\frac{4}{3}\gamma t^{\gamma-2}=\frac{2\Omega_{bdm}}{3}t^{\gamma-2}$$
$$\gamma^2-\gamma+\frac{4}{3}\gamma=\frac{2\Omega_{bdm}}{3}$$
$$\gamma^2+\frac{1}{3}\gamma-\frac{2\Omega_{bdm}}{3}=0$$

Baryonic Structure Formation

Linear Collapse and Jeans Length

To determine what happens to baryonic fluctuations we must incorporate both the effects of gravity and of pressure. Using Euler's equation and conservation of mass, we can derive the de for mass perturbations:

$$\frac{\partial^2\delta_k}{\partial t^2}+2H(t)\frac{\partial\delta_k}{\partial t}=\left(4\pi G\rho_b-\frac{k^2C_s^2}{a^2}\right)\delta_k$$

Where $\rho_b=\rho_0 a^{-3}$ is the average baryon density, $C_s=\frac{dP}{d\rho}$ is the speed of sound, and δ_k is the fluctuation in energy density at the wavenumber k (after Fourier decomposition).

This is a forced (by gravity-pressure), damped (by expansion) harmonic oscillator. The forcing and damping terms on the RHS cancel out for a special scale of perturbation called the Jeans wavelength. For $\lambda>\lambda_J$, the forcing effect of gravity dominates, while for $\lambda<\lambda_J$ the force of pressure dominates.

$$\lambda_J=\frac{2\pi a}{k_J}=\left(\frac{\pi C_s^2}{G\rho_b}\right)^{\frac{1}{2}}$$

Since the Jeans wavelength is a distance, we can define a corresponding Jeans mass which refers to the mass of a sphere the size of the Jeans radius and with the average baryon density:

$$M_J=\frac{4}{3}\pi\left(\frac{\lambda_J}{2}\right)^3\rho_b=\left(\frac{375k_B^3}{4\pi m^4G^3}\right)^{\frac{1}{2}}\left(\frac{T^3}{n}\right)^{\frac{1}{2}}\approx 3\times10^4\left(\frac{T^3}{n}\right)^{1/2}M_{solar}$$

Where m is the mass of a particle comprising the gas. This sort of analysis carried out for both baryonic and dark matter leads to a set of coupled equations describing the overdensities for each type of energy:

$$\frac{\partial^2\delta_k^{dm}}{\delta t^2}+2H(t)\frac{\partial\delta_k^{dm}}{\partial t}=4\pi G\left(\rho_{dm}\delta_k^{dm}+\rho_b\delta_k^b\right)-\frac{k^2C_s^2}{a^2}\delta_k^b$$
$$\frac{\partial^2\delta_k^{b}}{\delta t^2}+2H(t)\frac{\partial\delta_k^{b}}{\partial t}=4\pi G\left(\rho_{dm}\delta_k^{dm}+\rho_b\delta_k^b\right)$$

A simple way to decouple these equations is to assume that the dark matter dominates the gravity contribution:

$$\frac{\partial^2\delta_k^{dm}}{\delta t^2}+2H(t)\frac{\partial\delta_k^{dm}}{\partial t}=4\pi G\rho_{dm}\delta_k^{dm}-\frac{k^2C_s^2}{a^2}\delta_k^b$$

$$\frac{\partial^2 \delta_k^b}{\delta t^2} + 2H(t)\frac{\partial \delta_k^b}{\partial t} = 4\pi G \rho_{dm} \delta_k^{dm}$$

Solving these in an Einstein-De Sitter universe we find:

$$\delta_b = \frac{\delta_{dm}}{1 + Ak^2}$$

Where $A = \frac{3}{2}(\frac{k_B T_0}{m_p}\frac{t^2}{a^3})$ determines when pressure becomes important. Specifically, pressure becomes dominant at scales $\frac{1}{k} = \sqrt{A}$. m_p is the proton mass, and T_0 is the temperature when collapse begins. These compare to the matter-dominated case where:

$$\delta \propto t^{2/3}$$

On scales below the Jeans length, particles are unable to cluster, and hence densities do not grow over time. On large scales, neutrinos act like dark matter. On small scales, they add to the expansion rate (since they have energy), but do not contribute to gravity, so they perturbations grow more slowly.

Non-Linear Collapse and Virialisation

When baryonic mass virialises, it means that its energy terms are related by the virial theorem:

$$U = -2KE$$

Kinetic energy (KE) is related to temperature, which in turn is related to rms velocity by (using $m = \mu m_p$ for $\mu = \text{atomic weight} = 0.57 m_H$):

$$KE = \frac{1}{2}\sigma_v^2 m = \frac{3}{2}k_B T$$

Equation of state:

$$P = \frac{\rho k T}{M}$$

Temperature and hence velocity is expected to be the same at each radii assuming a singular isothermal sphere, thereby leading to the density profile for a galaxy being given by the equation:

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}$$

Solving the KE/v equation we find that the viralisation temperature is given by:

$$T_{vir} = 2.3 \times 10^5 (1+z)\left(\frac{M}{10^{12} M_{solar}}\right)^{2/3} h_{50}^{2/3}\ K$$

This is the temperature of a galaxy in virial equilibrium. A halo which ceases to collapse and reaches virial equilibrium is said to have virialised or to have thermalised (as net kinetic energy is converted to diffuse thermal kinetic energy).

Binding energy (or potential energy) of the gas in a galaxy is given by:

$$E \approx \frac{GM^2}{r}$$

For a sphere we can write more specifically:

$$V = -\frac{3}{5}\frac{GM^2}{r}$$

The virial radius is the radius of the galaxy, or the radius within which the viral theorem holds. It is usually approximated as:

$$r_{vir} = r(\rho = 180\rho_c)$$

This also allows us to define the virial mass as:

$$M_{vir} = \frac{4}{3}\pi r_{vir}^3(180\rho_c)$$

Another useful virial relation:

$$M \propto v_{vir}^3(1+z)^{-\frac{3}{2}}$$

Cooling Processes

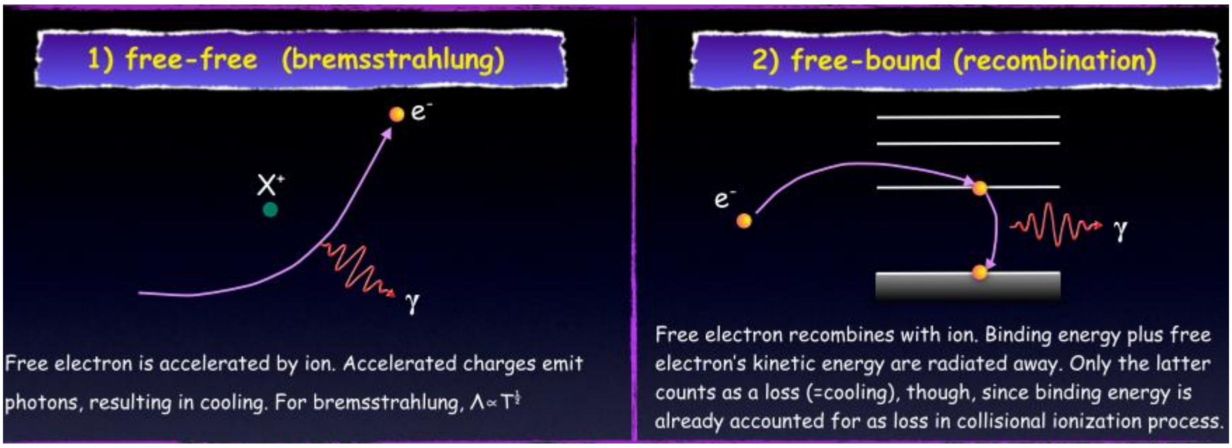
Baryonic perturbations typically begin with too much kinetic energy for the virial theorem to hold. As such, they must lose some kinetic energy before they can virialise, and continue to lose some energy in order to maintain virial equilibrium (as U increases over time due to gravitational collapse). The cooling time scale is given by:

$$\frac{E}{\dot{E}} \sim \frac{\frac{3}{2}k_B T\rho V/\mu m_p}{n^2\Lambda(T)V}$$

Where $\Lambda(T)$ is the energy loss per volume per time per density squared, and is called the "cooling function".

The two dominant cooling processes are Bremsstrahlung radiation and recombinations:

$$\epsilon_B = 1.4 \times 10^{-27} T^{\frac{1}{2}} (n_p x_e)^2$$
$$\epsilon_C = 7.5 \times 10^{-19} n_p^2 (x_e(1 - x_e)) \exp\left[-\frac{E_0}{k_B T}\right]$$



Where $E_0 = 13.6\ eV$ and x_e is the ionisation fraction of hydrogen atoms. We find x_e as:

$$x_e = \frac{1}{1 + a_{\mathrm{bohr}} \left(\frac{T}{10^5}\right)^{-7/6} \exp\left[\frac{E_0}{k_B T}\right]}$$

This allows us to define the cooling function as:

$$\Lambda(T) = \frac{\epsilon_B + \epsilon_C}{n_e^2} = 10^{-24} \frac{2.1\left(\frac{T}{10^5}\right)^{-7/6} + 0.44\left(\frac{T}{10^5}\right)^{\frac{1}{2}}}{\left(1 + a_{\mathrm{bohr}}\left(\frac{T}{10^5}\right)^{-7/6} \exp\left[\frac{E_0}{k_B T}\right]\right)^2}$$

Using this we can calculate the cooling time:

$$t_{cool} = \frac{E}{\dot{E}} = \frac{3\rho k_B T}{2\mu\Lambda(T) n_e^2}$$

Where the atomic mass is given by:

$$\mu = \frac{m_H n_H + m_{He} n_{He}}{2n_H + 3n_{He}}$$

If $t_{cool} > \frac{1}{H}$ then the cloud will never cool in the life of the universe.

If $t_{dyn} < t_{cool} < \frac{1}{H}$ then the cloud will undergo slow collapse. $t_{dyn} \sim \frac{R_{vir}}{\sigma_v} \sim 0.1H^{-1}$ is the time taken for free-fall collapse of the galaxy from the virial radius in the absence of pressure forces.

If $t_{cool} < t_{dyn}$ then the cloud undergoes direct and rapid collapse, fragmenting into smaller clusters which form stars. Galaxies always fall into this class.

Modified Press-Schechter Mass Function

Cooling affects the rate at which galaxies form and hence will modify the Press-Schechter mass function. Specifically, we see that for small T:

$$t_{cool} \propto T^{-\frac{1}{6}} \exp\left[\frac{2E_0}{k_B T}\right]$$

While for large T:

$$t_{cool} \propto T^{\frac{1}{2}}$$

This means that both for large and small T, t_{cool} is large, and therefore Press-Schechter will overestimate the numbers of galaxies formed at large and small masses.

It turns out that the necessary correction is simply to retain the same functional form, but correct the critical density as follows:

$$\delta_c(z_{col}) = 1.69(1 + z_{col})$$

Using the relation for a matter-dominated flat universe:

$$t_{cool} = t(z) - t(z_{col}) = \frac{2}{3}\frac{1}{H_0}\left((1+z)^{-\frac{3}{2}} - (1+z_{col})^{-\frac{3}{2}}\right)$$

For a galaxy to be formed at z, the halo must collapse at z_{col}.

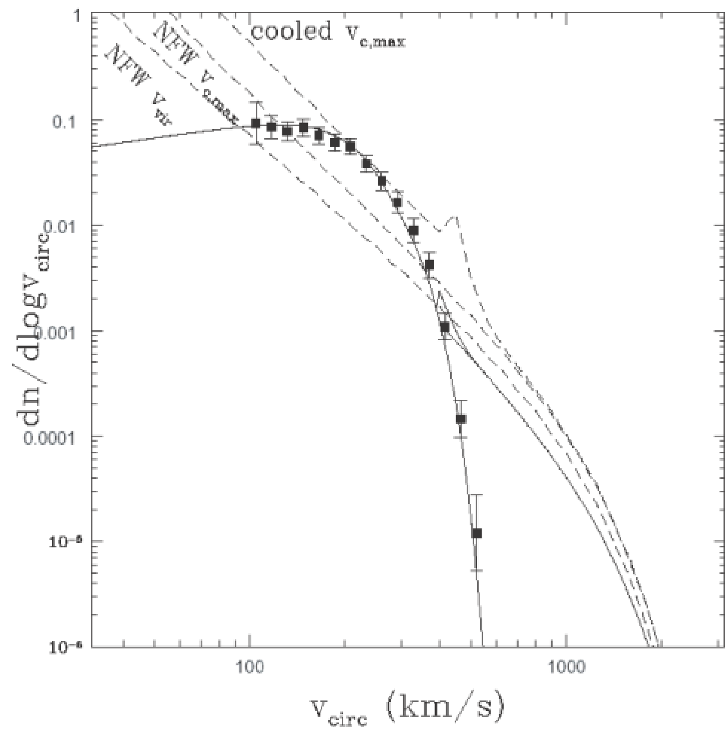


Figure 24: Comparison of Press-Schechter and simulation, from Kockanek and White, 2001, Figure 7.[3] "The velocity function $dn/d\log v_c = (dn/d\log M)|d\log M/d\log v_c|$. The solid curves show the local velocity function of galaxies (low v_{circ}) and clusters (high v_c) and their sum. The points are the non-parametric velocity function of galaxies. From bottom to top, the dashed curves show the velocity functions derived using dn/dM and the NFW virial velocity (labeled NFW v_{vir}), the peak circular velocity of the NFW rotation curve (labeled NFW $v_{c,max}$) and the peak circular velocity of the adiabatically compressed model (labeled cooled $v_{c,max}$). We used the $\Omega_{b,cool} = 0.018$ model with no bulge."

Origin of Angular Momentum

If a galaxy is rotationally supported, then the following relation will hold (for v circular velocity):

$$v^2(r) = \frac{GM}{r}$$

By the definition of angular velocity $\omega = \frac{2\pi v}{r}$ we find that the critical angular velocity for rotational support is:

$$\omega_{sup} = \frac{2\pi\sqrt{GM}}{r^{3/2}}$$

The angular momentum of such a galaxy is given by:

$$L = Mrv_{obs}$$
$$\omega_{obs} = \frac{2\pi v_{obs}}{r}$$

The spin parameter of a galaxy defines how close it is to having enough angular velocity to be rotationally supported. It is defined as:

$$\lambda = \frac{\omega_{obs}}{\omega_{sup}}$$

$$\lambda = \frac{L}{r\sqrt{ME}}$$

Using the expression for $E = GM^2/r$ we can substitutue out for r to find:

$$\lambda = \frac{L\sqrt{E}}{GM^{5/2}}$$

The angular momentum L initially derives from non-spherical perturbations in the density field, which is then amplified after virialisation as gravitational collapse continues to increase binding energy (as mass falls lower in the potential well), but L is conserved, thereby increasing λ.

The torque of one perturbation of oversensity δ on another is given by the product of acceleration, mass, and distance. Acceleration is given for commoving distance R:

$$A = \frac{G\delta M}{(aR)^2}$$

Thus we have an expression for the torque as :

$$T_r = \frac{G\delta M}{(aR)^2}\frac{M}{2}\frac{Ra}{2} = \frac{\delta GM^2}{4Ra}$$

This results in an expression for angular momentum of (for t time since beginning of universe):

$$L = T_r t = \frac{\delta GM^2}{4Ra}t$$

And hence an equation for λ is:

$$\lambda = \frac{L\sqrt{E}}{GM^{5/2}} = \frac{\delta}{a}\frac{M^2}{R}\frac{M}{\sqrt{Ra}}\frac{t}{M^{5/2}}$$

Using the virial relations $R \propto M^{\frac{1}{3}}$, $t_{col} \propto a^{\frac{3}{2}}$ and $\delta \propto a$, we can simplify this to:

$$\lambda \propto a_{col}M^{1/3}$$

Numerical simulations give this a value of around 0.025, indicating that a virialised halo is not rotationally supported. The value of λ then increases during cooling.

Size of Galaxies

Cooling of the baryons allows them to sink into the potential down to a radius where they are rotationally supported ($\lambda \sim 0.05 \rightarrow 1.0$). We can write the ratio of rotating disk to virialised halo spin parameters as:

$$\frac{\lambda_d}{\lambda} = \frac{L_d}{L}\left(\frac{E_d}{E}\right)^{\frac{1}{2}}\left(\frac{M_d}{M}\right)^{-\frac{5}{2}}$$

Using the following, where $k \sim 1$ are constants depending on the mass distribution, we find for a disk d and halo just at point of virialising h:

$$\frac{R_h}{R_d} = \left(\frac{M_d}{M_h}\right)\left(\frac{\lambda_d}{\lambda_h}\right)^2$$

$$\frac{M_d}{M} = \frac{\Omega_b}{\Omega_m} \approx 0.1$$
$$\frac{\lambda_d}{\lambda} = \frac{0.5}{0.025}$$
$$\frac{L_d}{L} = \frac{M_d}{M}$$
$$\frac{E_d}{E} = \frac{k_2}{k_1}\left(\frac{M_d}{M}\right)\left(\frac{R_d}{R}\right)^{\frac{1}{2}}$$

We find:

$$\frac{R}{R_d} \approx 40$$

Meaning that collapse to a rotationally supported system leads to a disk that is 40 times smaller than the virial radius of the halo.

In response to baryon cooling, the surrounding dark matter halo also shrinks in volume, effectively following the moving mass. For a singular isothermal sphere model with $\rho \propto \frac{1}{r^2}$ and $M(<r) = Ar$ with fraction α of baryons, we use the conservation of matter to find:

$$M_b(<r) + M_{dm}(<r) = Ar$$
$$rM_b(<r) + rM_{dm}(<r) = Ar^2$$
$$rM_b(<r) + rM_{dm}(<r) = \frac{M_{dm}^2(<r)}{(1-\alpha)^2 A}$$
$$M_{dm}(<r) = \frac{(1-\alpha)^2}{a} Ar\left(1 \pm \sqrt{1 + \frac{4M_b(<r)}{(1-\alpha)^2 Ar}}\right)$$

Since the initial dark matter profile is $M_i^{dm}(<r) = Ar(1-\alpha)$ and given $\alpha \ll 1$, we can write for the case when baryons collapse fully ($M_b(<r) = M_b = \alpha A r_{vir}$):

$$M_{dm}(<r) \approx M_i^{dm}(<r)\left((1-\alpha) + \frac{\alpha}{1-\alpha}\frac{r_{vir}}{r}\right)$$

This will represent a contraction so long as:

$$\frac{r}{r_{vir}} < \frac{1}{1-\alpha} < 1$$

An individual dark matter particle will have an orbital radius given by:

$$r = r_i\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4\alpha r_{vir}}{r}}\right)^{-1}$$

Indicating that dark matter particles are always drawn to the center as a result of baryon cooling.

Galaxy Clustering

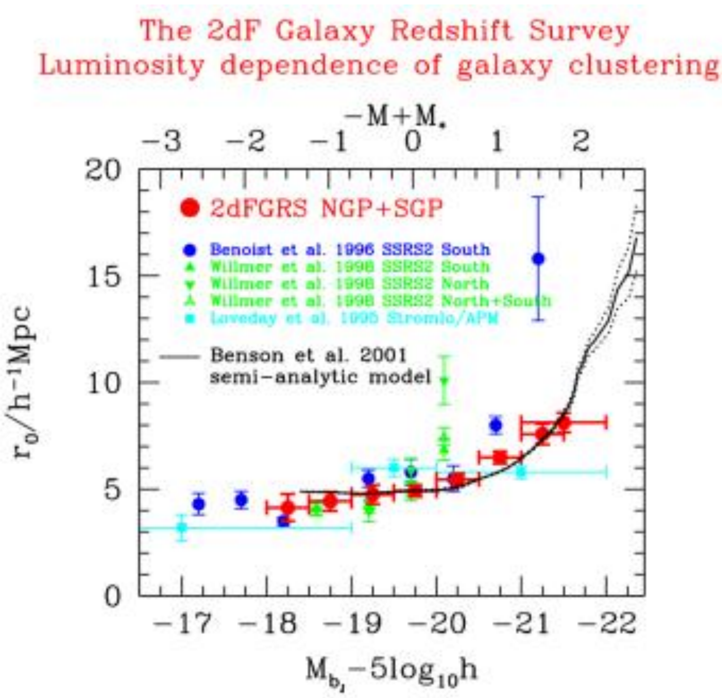
Galactic clustering is measured by a spatial correlation function of fractional overdensities:

$$\xi(r_1, r_2) = \langle \delta(r_1)\delta(r_2) \rangle$$

It is usually written as a power law:

$$\xi(r = r_1 - r_2) = \left(\frac{r}{r_0}\right)^{-\gamma}$$
$$\xi(r) \approx \left(\frac{r}{5\ h^{-1}\mathrm{Mpc}}\right)^{-1.8}$$

These two parameters allow us to compare the clustering of different galaxy populations: Increased $r_0$means stronger clustering, while increased γ means a steeper profile. As the results below indicate, big galaxies are more strongly clustered, i.e. have a larger scale length.



The same information can be encoded in the density fluctuation power spectrum:

$$P(k) = \int \xi(r) e^{ikr}\, dr^3$$

Which we can also write in dimensionless form:

$$\Delta^2(k) = \frac{k^3}{2\pi^2} P(k)$$

Because of galaxy clustering, we do not expect galaxy distribution to accurately represent the underlying mass density field. We assume that there is a linear bias between the two:

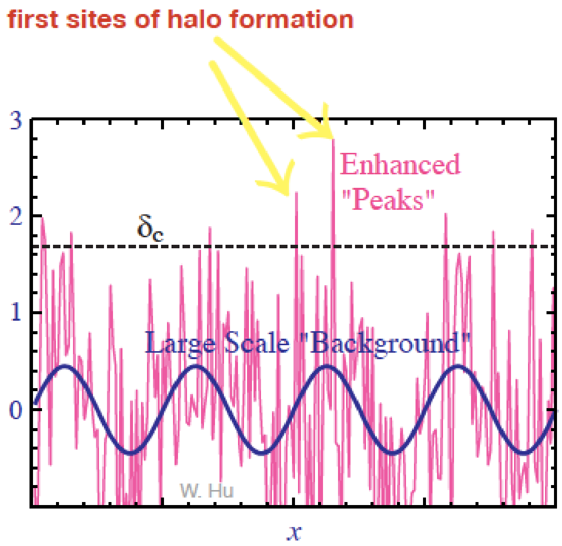
$$\delta_{gal} = b\delta_{mass}$$
$$P_{gal} = b^2 P_{mass}$$
$$\xi_{halo} = b^2 \xi_{DM}$$

The bias is defined as:

$$b = \frac{\delta_{gal}}{\delta_{mass}}$$

Where the subscripts refer to the galaxy and underlying mass distribution respectively. Note that these values are redshift dependent, so need to be converted to a common redshift for comparisons:

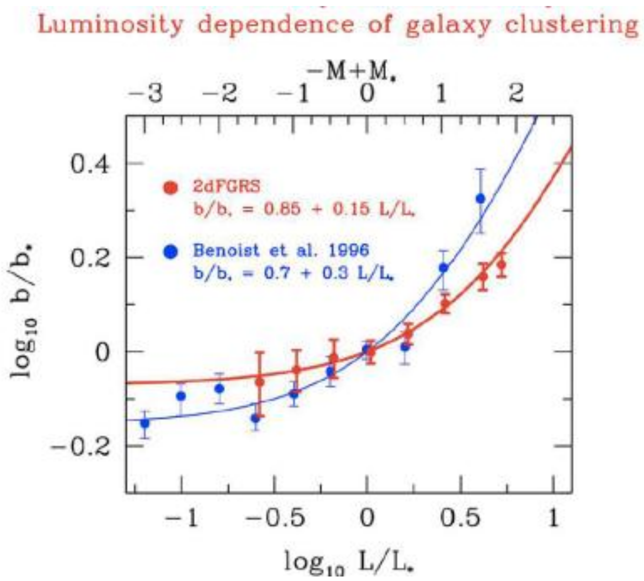
$$\delta_{z_0} = \frac{\delta_0}{1 + z_0}$$



In the Press-Schechter formalism we can write:

$$b = \left(1 + \frac{v^2 - 1}{v\sigma(M)}\right)$$
$$v = \frac{\delta_c - \delta}{\sigma(M)}$$

We can try to measure bias using HOD – halo occupation distribution, which is how many galaxies reside in each dark matter halo. This is complicated by the fact of galactic mergers.



Neutrino Freezout and Nucleosynthesis

Key Thermodynamics Relations

The following relations depend upon the distribution function over phase-space $f(\tilde{p})$ and the degeneracy factor g.

The number density is:

$$n = \frac{g}{(2\pi)^3} \int f(\tilde{p})\, dp^3$$

The energy density is:

$$\rho = \frac{g}{(2\pi)^3} \int f(\tilde{p}) E(\tilde{p})\, dp^3$$

The pressure is:

$$P = \frac{g}{(2\pi)^3} \int f(\tilde{p}) \frac{p^2}{3E(\tilde{p})} dp^3$$

In thermal equilibrium the distribution function has the form:

$$f(\tilde{p}) = \frac{1}{\exp\left[\frac{E-\mu}{T}\right] \pm 1}$$

Outside of thermal equilibrium, the Boltzmann equations must be used.

Some other useful relations:

$$E(p) = \sqrt{p^2 + m^2}$$
$$KE = \frac{1}{2}\sigma_v^2 m = \frac{3}{2} k_B T$$
$$P = \frac{\rho k T}{M}$$

High and Low Energy Limits

For the high temperature limit valid in the very early universe, when everything is relativistic, we have:

Bosons

$$\rho = \left(\frac{\pi^2}{30}\right) g T^4$$
$$n = \left(\frac{\zeta(3)}{\pi^2}\right) g T^3$$
$$P = \frac{\rho}{3}$$

Fermions

$$\rho = \frac{7}{8}\left(\frac{\pi^2}{30}\right) g T^4$$
$$n = \frac{3}{4}\left(\frac{\zeta(3)}{\pi^2}\right) g T^3$$
$$P = \frac{\rho}{3}$$

We can also write down the mean energy per particle:

$$\langle E \rangle = \rho/n = 2.701\, T \text{ (boson) or } 3.151\, T \text{ (fermion)}$$

For the low temperature limit in later times when temperature falls below particle mass and the particles are non-relativistic, we have:

$$\rho = nm$$
$$n = g\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{(m-\mu)}{T}}$$
$$P = nT$$

We note from these equations that, in thermal equilibrium the number of non-relativistic particles is exponentially suppressed, and thus they are comparably unimportant. Today, of course, the universe is not in thermal equilibrium, so now the universe is dominated by non-relativistic particles.

Entropy

In thermal equilibrium (and thus high temperature), the entropy per commoving volume is constant:

$$S = \frac{a^3(\rho + P)}{T}$$
$$= a^3 \frac{\left(\frac{\pi^2}{30}\right) g_* T^4 + \frac{1}{3}\left(\frac{\pi^2}{30}\right) g_* T^4}{T}$$
$$= a^3 \frac{4}{3}\left(\frac{\pi^2}{30}\right) g_* T^3$$
$$S = a^3 \left(\frac{2\pi^2}{45}\right) g_* T^3$$

Where g_* is the new effective degree of freedom:

$$g_* = \sum_{i \in \text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{j \in \text{fermions}} g_j \left(\frac{T_j}{T}\right)^3$$

The number of relativistic degrees of freedom affects the expansion rate (in a flat, radiation-dominated universe) by:

$$H \propto \sqrt{g^*} \frac{T^2}{M_{pl}}$$

For example, if we have photons, electrons, and neutrinos, we would use:

$$g_* = 2 + \left(\frac{7}{8}\right)(4 + 6) = 10.75$$

To see what happens at neutrino/photon decoupling, we use conservation of entropy:

$$10.75 a_0^3 \left(\frac{2\pi^2}{45}\right) T_0^3 = \frac{42}{8}\left(\frac{2\pi^2}{45}\right) T_\nu^3 a_1^3 + 2\left(\frac{2\pi^2}{45}\right) T_1^3 a_1^3$$

Let's set T_0 and a_0 to 1, resulting in:

$$10.75 = \frac{21}{4} T_\nu^3 a_1^3 + 2 T_1^3 a_1^3$$

We also know that for the neutrinos we must have $T_0^3 a_0^3 = T_\nu^3 a_1^3$, or $T_\nu^3 = a_1^{-3}$:

$$10.75 = \frac{21}{4} + 2T_1^3 T_\nu^{-3}$$
$$\left(\frac{T_1}{T_\nu}\right)^3 = \frac{11}{4}$$
$$T_\nu = \left(\frac{4}{11}\right)^{\frac{1}{3}} T_\gamma$$

This prediction, however, will only hold if neutrinos are still in a thermal equilibrium distribution with each other. Since decoupling with photons, however, neutrinos have started to become non-relativistic (owing to their non-zero mass), and will therefore no longer follow a thermal distribution.

Freezout

Freezeout occurs when a particular reaction rate drops too low to maintain equilibrium in an expanding Universe. Thus it depends on how quick the reaction is and how quickly the Universe is expanding. Equilibrium can only be preserved if there are frequent enough interactions to maintain the momentum and density distributions. Generally the rigorous way to define this is that there must be an expectation of at least one interaction in the age of the universe. This we say that a particular reaction freezes out when the time between reactions $1/\Gamma$ and age of the universe $1/H$ are of the same size:

$$\frac{\Gamma}{H} = 1$$

Two-particle interactions typically have $\Gamma = n\sigma v$. Using the relation:

$$H \propto \sqrt{g^*}\frac{T^2}{M_{pl}}$$

In general, the reaction rate scales as $\Gamma \propto T^n$. We can write (for relativistic particles):

$$\sqrt{g^*}\frac{T^2}{M_{pl}} = \langle\sigma v\rangle T^3$$
$$T = \frac{\sqrt{g^*}}{M_{pl}\langle\sigma v\rangle}$$

Massive particles maintain thermal equilibrium through scattering and annihilation reactions so long as $T \gg m$. Freezing out, or decoupling, can occur at relativistic or non-relativistic temperatures depending on the mass of the particle. When these events occur determines the ratio of the resulting particles.

For example, we can find the photon-baryon ratio using:

$$\rho_b = n_b m_b$$
$$n_b = \frac{\rho_b}{m_b \rho_c}\rho_c$$
$$n_b = \frac{\Omega_b \rho_c}{m_b}$$
$$\frac{n_b}{n_\gamma} = \frac{\rho_c}{n_\gamma m_b}\Omega_b = 2.7 \times 10^{-8}\Omega_b h^2$$

This implies that there are many photons for each baryon.

Recombination

Recombination is a particular instance of freeze-out, referring to the time when photons decouple from electrons and protons, leading to the first formation of neutron atoms (mostly hydrogen). When this occurs, the number of free electrons drastically falls. Once decoupling has occurred, it is unlikely that a photon will ever interact again in the age of the universe, and so the photons free-stream to us in the present. The image we see from these photons is called the cosmic microwave background.

Now we will estimate the timing of recombination. We will assume thermal equilibrium, even though this obviously isn't an equilibrium process since it exactly involves the end of equilibrium. Let number densities be denoted with n_e, n_p, n_H, and let n be the density of protons plus hydrogen. By charge conservation $n_e = n_p$. The ionisation fraction as $x_e = \frac{n_e}{n} = x_p = 1 - x_H$. In the non-relativistic limit, the number density for species a becomes:

$$
\begin{aligned}
n_a &= \frac{g_a}{2\pi^2}\int p^2 e^{\frac{\mu-E}{T}}\,dp \\
&= \frac{g_a}{2\pi^2}\int p^2 e^{\frac{\mu-\sqrt{p^2+m^2}}{T}}\,dp \\
&= \frac{g_a}{2\pi^2}\int p^2 e^{\frac{\mu-m\sqrt{1+(p/m)^2}}{T}}\,dp \\
&\approx \frac{g_a}{2\pi^2}\int p^2 e^{\frac{\mu-m\left(1+\frac{p^2}{2m}\right)}{T}}\,dp \\
&= \frac{g_a}{2\pi^2}\int p^2 e^{\frac{\mu-m}{T}} e^{-\frac{p^2}{2mT}}\,dp \\
&= \frac{g_a}{2\pi^2} e^{\frac{\mu-m}{T}}\int p^2 e^{-\frac{p^2}{2mT}}\,dp \\
&= \frac{g_a}{2\pi^2} e^{\frac{\mu-m}{T}}\left[\sqrt{\frac{\pi}{2}}(mT)^{\frac{3}{2}}\right] \\
n_a &= g_a\left(\frac{mT}{2\pi}\right)^{3/2} e^{\frac{\mu-m}{T}}
\end{aligned}
$$

We thus have three equations:

$$
\begin{aligned}
n_e &= 2\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{\frac{\mu_e-m_e}{T}} \\
n_p &= 2\left(\frac{m_p T}{2\pi}\right)^{3/2} e^{\frac{\mu_p-m_p}{T}} \\
n_H &= 4\left(\frac{m_H T}{2\pi}\right)^{3/2} e^{\frac{\mu_H-m_H}{T}}
\end{aligned}
$$

Multiplying the first two equations we have:

$$
n_e n_p = 4\left(\frac{m_e T}{2\pi}\right)^{3/2}\left(\frac{m_p T}{2\pi}\right)^{3/2} e^{\frac{\mu_e-m_e+\mu_p-m_p}{T}}
$$

From the thermal equilibrium reaction $p + e \to H + \gamma$ we get the relation $\mu_p + \mu_e = \mu_H$. Using this and electrical neutrality we have:

$$
n_p^2 = 4\left(\frac{m_e T}{2\pi}\right)^{3/2}\left(\frac{m_p T}{2\pi}\right)^{3/2} e^{\frac{\mu_H-m_e-m_p}{T}}
$$

Substituting in the third equation and using $m_p \approx m_H$ we find:

$$n_p^2 = \left(\frac{m_e T}{2\pi}\right)^{3/2} n_H e^{\frac{m_p + m_e - m_H}{T}}$$

The exponential term is the binding energy of a hydrogen atom, equal to 13.6 eV.

$$n_H = n_p^2 \left(\frac{m_e T}{2\pi}\right)^{-3/2} e^{\frac{13.6 \text{ eV}}{T}}$$

We define the ionization fraction as $x_e = \frac{n_p}{n} = \frac{1-n_H}{n}$, yielding:

$$n - n x_e = n^2 x_e^2 \left(\frac{m_e T}{2\pi}\right)^{-3/2} e^{\frac{13.6 \text{ eV}}{T}}$$
$$\frac{1 - x_e}{x_e^2} = n \left(\frac{m_e T}{2\pi}\right)^{-3/2} e^{\frac{13.6 \text{ eV}}{T}}$$

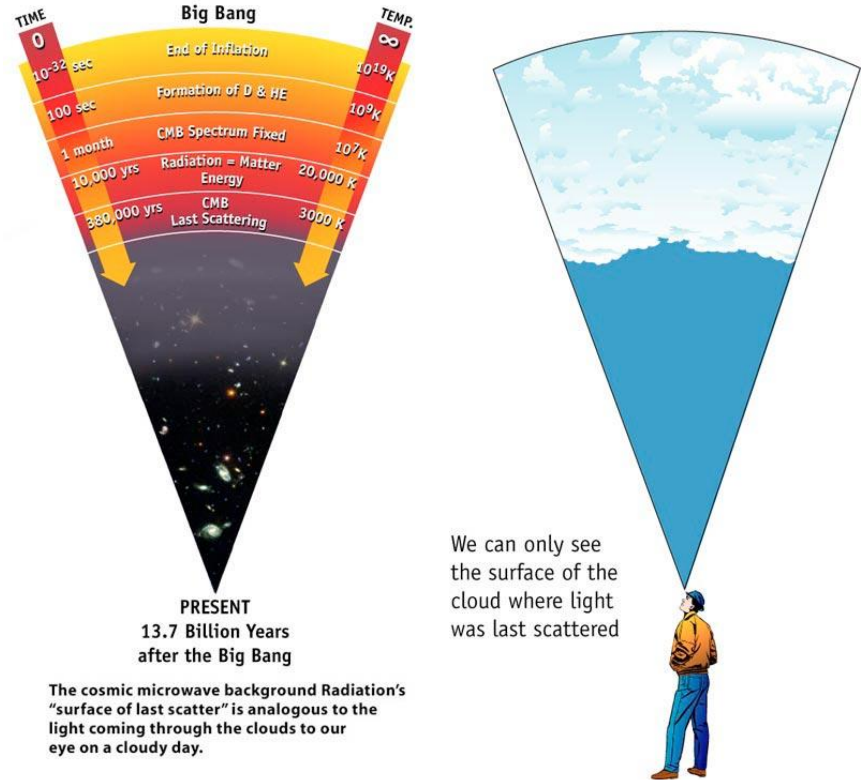
This is known as the Saha equation, and gives the equilibrium ionisation fraction as a function of temperature. Solving it indicates that 90% recombination occurs by T=0.3 eV, a redshift of around 1260. An alternate form of the equation is given in terms of the photon-baryon ratio η:

$$\frac{1 - x_e}{x_e^2} = \eta \left(4\zeta(3)\sqrt{\frac{2}{\pi}}\right)\left(\frac{T}{m_e}\right)^{3/2} e^{-B/T}$$

The Saha equation is only valid so long as the reaction $e + p \rightarrow H$ is fast compared to the expansion of the universe, which obviously fails near recombination because this involves this reaction freezing out. To work out the real answer therefore it is necessary to solve the Boltzmann equations numerically. This calculation gives recombination occurring around $z = 1100$.

Sound Horizon at Surface of Last Scattering

Recombination is when electrons and protons combine to form neutral atoms, and marks the entering of the matter-dominated epoch. Photon decoupling is when photons cease interacting (at any appreciable rate) with the baryonic matter in the universe. Photon decoupling occurs around the same time as recombination, but somewhat afterwards owing to the much greater abundance of photons. The surface of last scattering is not precisely defined: the width of the LSS, ie the range of redshifts at which photons actually scatter last, is about Δz = 100.



The cosmic microwave background Radiation's "surface of last scatter" is analogous to the light coming through the clouds to our eye on a cloudy day.

The commoving distance to the sound horizon at the time of last scattering was:

$$r_s = \int_0^{\eta_{LS}} c_s \, d\eta$$

Using the fact that conformal time can be rewritten $d\eta = \frac{1}{a} dt$, we have:

$$\begin{aligned} r_s &= \int_0^{t_{LS}} c_s \frac{1}{a} \mathrm{d}t \\ &= \int_{z_{LS}}^{\infty} c_s \frac{1}{a} \frac{dt}{da} \frac{da}{dz} dz \\ &= \int_{z_{LS}}^{\infty} c_s \, (1 + \mathrm{z}) \left(\frac{1}{\dot{a}}\right) \frac{1}{(1+z)^2} dz \\ &= \int_{z_{LS}}^{\infty} c_s \left(\frac{a}{\dot{a}}\right) dz \\ r_s &= \int_{z_{LS}}^{\infty} \frac{c_s}{H} dz \end{aligned}$$

The speed of sound, with the ratio $R = \frac{P_b + \rho_b}{P_\gamma + \rho_\gamma}$, is given by:

$$c_s = \frac{1}{\sqrt{3(1+R)}}$$

We typically use $c_s \approx 0.5$. To calculate the angular size (in radians) of this, we use:

$$\theta_{sls} = \frac{r_s}{D_A} = \frac{r_s}{D_A(1+z)}$$

Where the extra factor of $1 + z$ is needed to convert commoving r_s into a proper physical distance. Angular frequency is found by:

$$\frac{l}{\pi} = \frac{1}{\theta_{sls}}$$

The WIMP Miracle

WIMPS are a hypothesised candidate for dark matter. If they exist, they would have begun (like everything) at relativistic temperatures and in thermal equilibrium with photons, leptons, and quarks. Eventually they cooled and became non-relativistic, and shortly thereafter the rate at which they annihilated froze out, thereby leaving some residual relic abundance of these massive, weakly interacting particles.

If the particle X has number density n with cross-section σ, then the rate of annihilation is:

$$\Gamma = \langle \sigma v \rangle \, n_X$$

The Boltzmann equation of relevance in this case has the form:

$$\frac{dn_X}{dt} = -3Hn_X + \frac{g}{(2\pi)^3} \int \frac{C[f]}{E} dp^3$$

We can understand this equation as saying the change in number density versus time equals the dilution term due to the Universe's expansion ($-3Hn$) and the effect of interactions that change the number of particle X's present ($\int \frac{C[f]}{E} dp^3$). If there are no further non-equilibrium coupled equations for other particles, this becomes:

$$\frac{dn_X}{dt} = -3Hn_X + < \sigma v > \left(n_X^2 - \left(n_X^{eq} \right)^2 \right)$$

With $H = \frac{5}{3} g_*^{\frac{1}{2}} \frac{T^2}{m_{pl}}$ since this occurs during radiation domination. n_X is the actual number density and n_X^{eq} is the equilibrium number density. The cross section $< \sigma v >$ refers to the annihilation reaction.

We will assume that decoupling occurs for $T_{dec} < m_x$, meaning that the particles are non-relativistic during decoupling. Between when the particle goes non-relativistic to when it decouples, it's number density will trace the exponentially falling equilibrium number density, at the rate $e^{-\frac{m-\mu}{T}}$.

This results in a predicted residual abundance of WIMPs of roughly:

$$\Omega_x = \frac{1}{< \sigma v > h^2}$$
$$\Omega_x = \left(\frac{< \sigma v >}{3 \times 10^{-27} \text{cm}^3/s} \right)$$

For the value of Ω_{dm} that we observe, $< \sigma v >$ would have to be 3×10^{-27} cm^3/s. The miracle is that this is very close to the sorts of reaction cross-sections predicted based on varies theories from particle physics for new weakly-interacting particles. Thus it seems that WIMPs are a very good dark matter candidate.

Big Bang Nucleosynthesis

At high temperatures well above nuclear binding energies, the thermodynamically favored state for baryons is a sea of free protons and neutrons, which maximizes entropy. As the temperature drops below the binding energy, the binding energy of the nucleus comes into play, and it becomes thermodynamically favored to bind p+n into nuclei. While the most tightly bound element is iron, the reactions needed to form it take time, longer than was available during the first few minutes during the Big Bang when the universe was the right temperature for these reactions to occur.

At early times ($t \ll s, T \gg MeV$), neutrons and protons are kept in thermal equilibrium by the reactions $n + \nu_e \leftrightarrow p + e$. The relative abundance of these particles will remain in thermal equilibrium as long as the rate of this reaction is rapid compared to the expansion of the universe. By the time of BBN temperatures are in the MeV range, well less than the 900MeV mass of protons, and so they follow the low energy limit. The equilibrium the neutron/proton ratio (ignoring the small chemical potential terms) is thus given by:

$$\frac{n}{p} = \exp\left[-\frac{m_n - m_p}{T}\right] = \exp\left[-\frac{1.293\ \text{MeV}}{T}\right]$$

From this we see that at high temperatures the n:p ratio approaches 1:1, while for low temperatures we have all protons. Thus to find the actual ratio we need to know when these reactions froze out, which occurred at a temperature of around $T_{dec} = 0.8$ MeV, yielding:

$$\frac{n}{p} = 0.2$$

By the time the universe has cooled to around 0.1 MeV, basically all of the neutrons have incorporated into Helium. So we find the helium mass fraction as:

$$Y_p = \frac{4n_{He}}{n_n + n_p} = \frac{4\left(\frac{n_n}{2}\right)}{n_n + n_p} = \frac{2(n_n/n_p)}{1 + n_n/n_p}$$

Using a value of $\frac{n}{p} = \frac{1}{7}$, slightly lower than the 0.2 found earlier owing to neutron decay, we find:

$$Y_p = 0.25$$

Using observations from very lower metallicity stars, we find observationally a value of around 0.2465, so theory and observations are in good agreement.

Baryogenesis

The process by which baryons are produced in the early Universe is known as baryogenesis. Freezeout theories discussed above do not work well for baryons, since they predict too much antimatter (you'd expect 50/50 from the general reaction $\gamma + \gamma \rightarrow q + \bar{q}$ so $\mu = 0$).

It also predicts far too few baryons, since baryons interact far more with photons than dark matter, we'd expect them to have remained in equilibrium longer than dark matter, and thus faced longer exponential suppression, leading to much lower prevalence today compared to dark matter, not a mere fact of five less.

Once the temperature drops below the mass of the proton/neutron at a GeV or so, the reaction starts running only one way; to the left. Annihilation dominates and the baryon number density begins to drop exponentially till the reaction freezes out, which occurs at around $T = \frac{m_p}{42} = 22\text{ MeV}$. We then calculate the baryon/photon ratio (for non-relativistic baryons) at this temperature to find:

$$\eta = \frac{n_b}{n_\gamma}$$
$$= \frac{g\left(\frac{m_b T}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{(m_b-\mu)}{T}}}{\left(\frac{\zeta(3)}{\pi^2}\right) g T^3}$$
$$= \frac{\sqrt{2\pi}}{\zeta(3)}\left(\frac{m_b}{T}\right)^{3/2} e^{-\frac{m_b}{T}}$$
$$\frac{n_b}{n_\gamma} \approx 10^{-18}$$

The observed ratio, by contrast, is about 10^{-9}, far more than anticipated.

The anticipated solution is that there must be a high energy process that does not preserve baryon number. If this is the case, then there will be some residual $N_b = n_b - n_{\bar{b}}$ which does not decay exponentially, but remains roughly constant, thereby yielding many more residual protons. At early times, this difference would be tiny, about one in a billion. But if for every billion antiquarks there were a billion and one quarks, that leaves us today with 1 quark and 1 billion photons.

The Sakharov conditions define three necessary conditions to generate this initial asymmetry:

1. Violate baryon number conservation, which is predicted by a number of GUT theories which also predict proton decay
2. C and CP violation (charge conjugation and charge parity). The former is violated by the weak force, the latter by various subatomic particles
3. Reaction for generating baryons/antibaryons runs out of thermal equilibrium (since otherwise with equal masses the reaction runs both ways and you get equal numbers of each)

Cosmic Microwave Background Radiation

Introduction

Recombination is when the initial hot plasma cools sufficiently that neutral Hydrogen forms. At this point, the photons are no longer trapped by frequent Thomson scattering and can free-stream to us today. The sea of these photons is called the cosmic microwave background or CMB. The image we're looking at represents the temperature fluctuations in the CMB black-body temperature as a function of position on the sky. The perturbations in the CMB are very small, only about one part in 100,000.

Power Spectrum

The spherical harmonic functions are the orthonormalised set of all functions over the domain of the surface of a sphere (so there are two input angle arguments), which satisfy Laplace's equation. Any function over a sphere can thus be written in terms of the spherical harmonic functions. The multipole number l determines how many 'regions' the sphere surface is divided into, so higher l numbers correspond to higher frequency fluctuations. The parameter m effectively determines the 'arrangement' of these modes, and varies as $-l \leq m \leq l$, so larger l values have more possible arrangements, as expected. The expected energy in each frequency mode l is given by the average over all the corresponding m values:

$$C_l = \frac{1}{2l+1} \sum_{m=-l}^{m=l} |a_{lm}|^2$$

Usually what is plotted is the squared amplitude of the temperature fluctuation, which is given by:

$$\mathcal{D}_l = T_0^2 \frac{l(l+1)}{2\pi} C_l$$

Over small angular distances, the spherical harmonic l equals the wavenumber k in radians:

$$l = |k| = \frac{2\pi}{\lambda}$$

Sources of Anisotropy

Intrinsic temperature variations

The plasma was actually hotter (or colder) along that line of sight at the last scattering surface. One way to do this would be an adiabatic compression, which heats an ideal gas. This will be directly related to gravitational overdensities of dark matter, which will pull in the photon-baryon fluid nearby. For this radiation dominated era of the universe:

$$\frac{\delta T}{T} = \frac{\delta \rho}{\rho}$$

Doppler shifts

There's no reason to expect that we are at rest with the local rest frame of the radiation in the early Universe. The relative velocity between the radiation rest frame and the observer rest frame induces a temperature variation that scales as v/c. If there is motion to the observer, the temperature increases. If away, the temperature decreases. We expect velocities to be present in the early Universe because there are dark matter over- and under-densities for the photon-baryon plasma to fall into (or out of).

Sachs-Wolfe effect

The generic Sachs-Wolfe effect occurs due to the gravitational redshift of photons. Say at a point x, the potential is zero, while at the observer the potential is positive. The travelling photon therefore experienced a net energy loss, causing a redshift and thus a cooler appearance.

The integrated Sachs-Wolfe effect is similar but occurs when gravitational potentials are evolving over time, causing loss or gain of energy during the process of transiting an energy potential well. The effect is absent in a matter-dominated universe with fixed potentials. It is present at very early (radiation dominant) and late (dark energy dominant) time periods. We can write down the combination of all these effects for observed temperature fluctuations along a line of sight n as:

$$\Theta_{obs}(n) = \left(\frac{\delta T}{T}\right)_{obs} = ITV + DS + SWE + ISWE$$

$$\Theta_{obs}(n) = \Theta - n \cdot v_b + \psi + \int_{t_{LSS}}^{t_0} (\dot{\psi} + \dot{\phi})\, dt$$

Reducing anisotropy

In addition to the three processes that generate or increase anisotropy, there are also two processes that tend to reduce anisotropy, or push the CMB back towards isotropy.

Optical depth

Reionisation produces free electrons which can scatter photons, thereby smearing out the anisotropy. On average the CMB photons have not interacted with matter since recombination (the LSS), however this isn't true in every single case, since there are free electrons around the cosmos. About 8% of CMB photons scatter off one of these free electrons before reaching us. The optical depth $\tau = 0.08$. This Thomson scattering will alter the incoming angle of the photon, thereby creating a blurring of anisotropy detail for higher l values. The power spectrum is reduced by $e^{-2\tau}$ on these small scales.

Silk damping

Photons diffuse (in a random walk) from the hot, overdense regions of plasma to the cold, underdense ones. The photons pushed electrons along, and these, in turn, pulled on protons by the Coulomb force. This caused the temperatures and densities of the hot and cold regions to be averaged and the universe became less anisotropic (characteristically various) and more isotropic (characteristically uniform). This effect is much larger for smaller-sized angular fluctuations, and hence dominates for large values of l.

Explaining the Power Spectrum

Acoustic oscillations

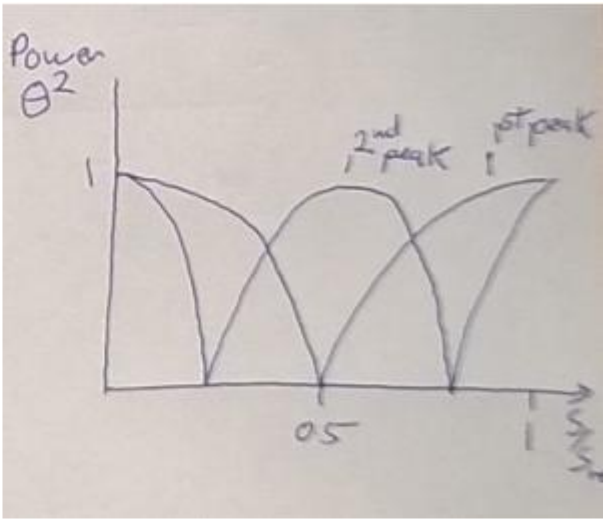
The oscillations came about in the following way: Regions of higher dark matter density cause increased gravitational attraction. In consequence, the plasma density increases in such regions as well: As plasma flows into the region, it gets compressed. Compressed plasma has a higher internal pressure, chiefly due to the photons but also transferred to the plasma itself via electromagnetic interaction. Once the pressure has increased sufficiently, it drives the plasma particles apart, leading to lower-than-average plasma density in that particular region. The plasma pressure drops as well, and thus, by its gravitational attraction, the dark matter can again pull more plasma into the region - thereby increasing the plasma density, and the cycle can start anew.

The resulting plasma oscillations are very similar to sound waves - sound waves, after all, are propagating, periodic density fluctuations in air. Therefore, physicists call these plasma oscillations "acoustic oscillations". The largest wavelength possible for these acoustic oscillations is the distance travelled by the photon-baryon plasma during the 400,000 years before recombination. This timing determines the location of the fundamental mode (highest peak) of the acoustic oscillations.

If s_{rec} is the commoving recombination horizon, then peaks will occur at:

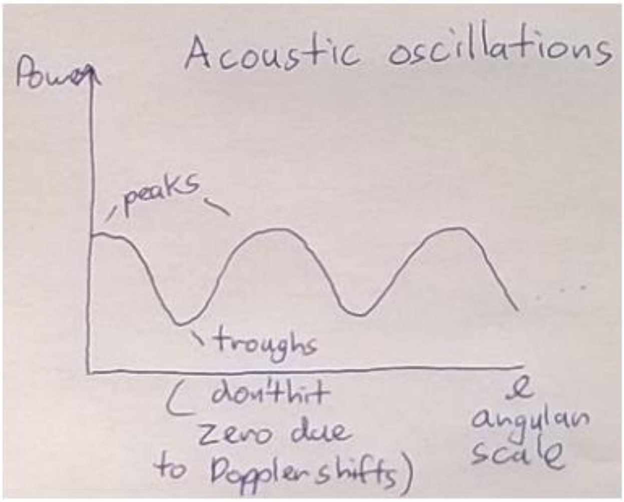
$$ks_{rec} = n\pi$$

In a flat universe this results in an initial peak at around $l = 200$.



Non-zero minimum

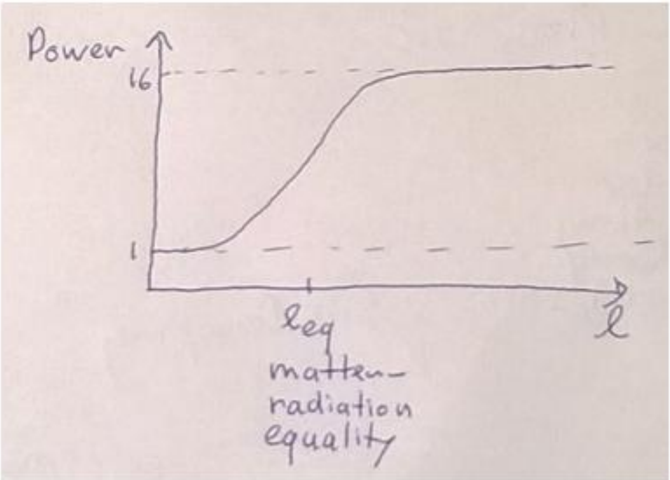
The fact that the sine and cosine acoustic oscillations never result in a zero power minimum is due to the effect of Doppler shift. There would be a Doppler shift term that would be 90 degrees out of phase with the temperature maxima, a result of the fact that the greatest velocity variations will occur in between modes of greatest temperature variations. If it was the same magnitude, we wouldn't see oscillations again since cosine squared plus sine squared is one. However the magnitude of the Doppler term is lower due to the directional dependence of the dopper effect.



Sachs-Wolfe plateau

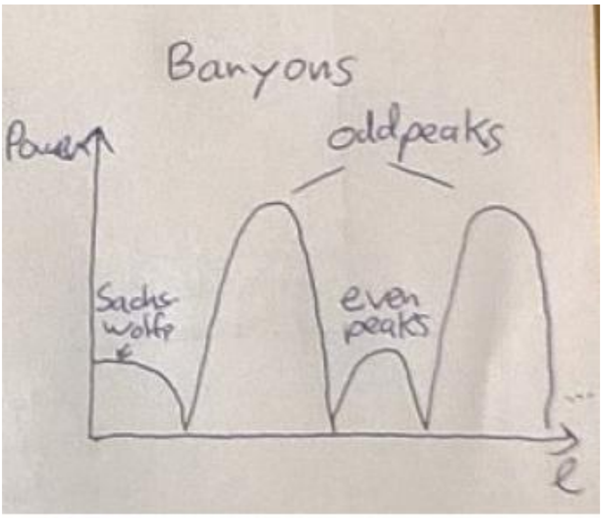
The leftmost low-level plateau is due to the fact that modes bigger than the fundamental length (the first peak) were larger than the universe at the time of recombination, and thus the temperature fluctuations were 'frozen' at this scale. They thus show the flat line (more or less) that we would expect given the scale-invariance of the initial fluctuations, absent the acoustic oscillations.

We also expect the plateau region to be only 1/16 the height of the subsequent regions. This is due to the fact that these higher l regions entered the horizon during the radiation-dominated era, and thus are not subject to the dampening effect of baryons (being massive) that occurs during the matter-dominated era. The actual effect is less than 16 because of exponential dampening.



Baryon loading

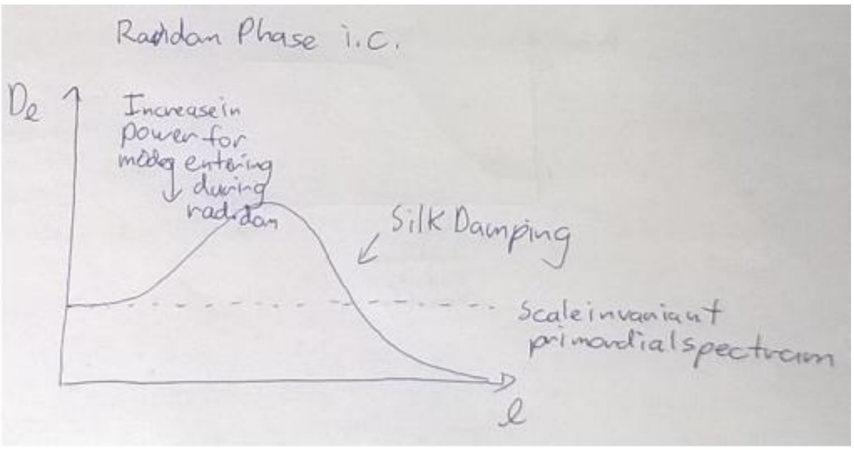
Baryons add inertia to the photon-baryon plasma; extra mass. This reduces the speed of sound according to the equation $c_s^2 = \frac{1}{3(1+R)}$, thereby shifting the position of the peaks along the l axis by a factor proportional to $\sqrt{1+R}$. The amplitude of the oscillations will also be increased by a factor of 1+3R. The most important effect of baryons, however, is that they cause an increase in the amplitude of odd peaks relative to even peaks. This is especially noticeable in the magnitude of the third peak, which should be a lot smaller than the second peak due to exponential damping, but is in fact roughly the same size.



Silk damping

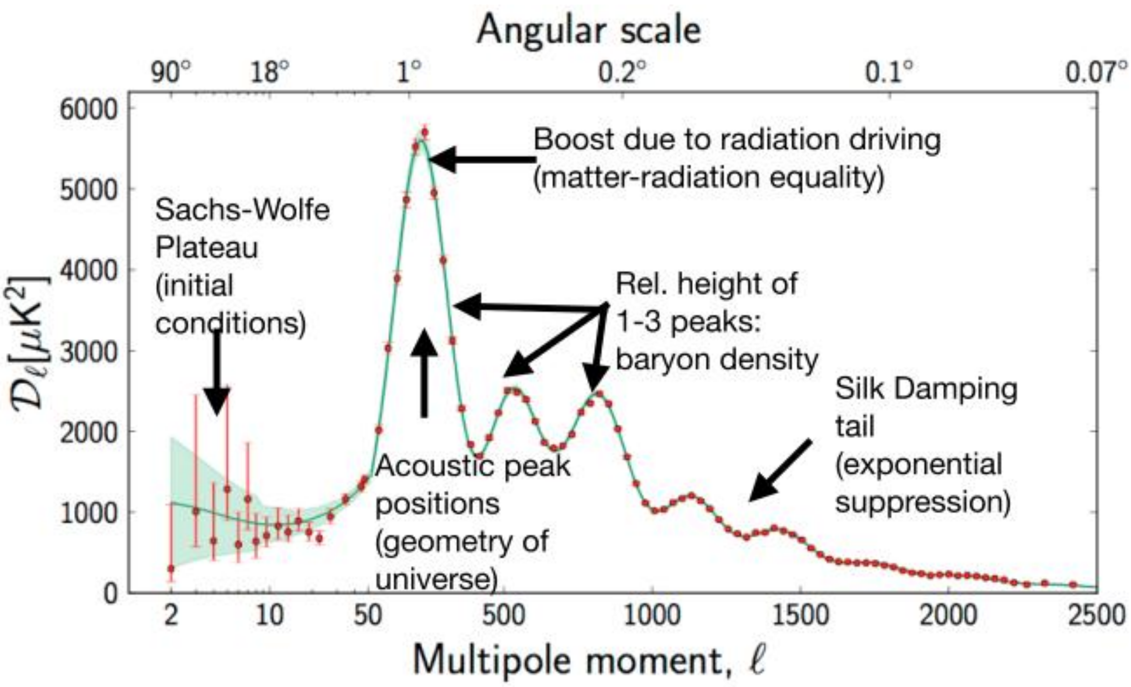
As described above, silk damping reduces the difference in temperature fluctuations owing to the diffusion of photons from hot to cold patches. The smaller the angular size of the fluctuations in question, the more such diffusion takes place, and thus the more the amplitude of the fluctuations at this scale is suppressed.

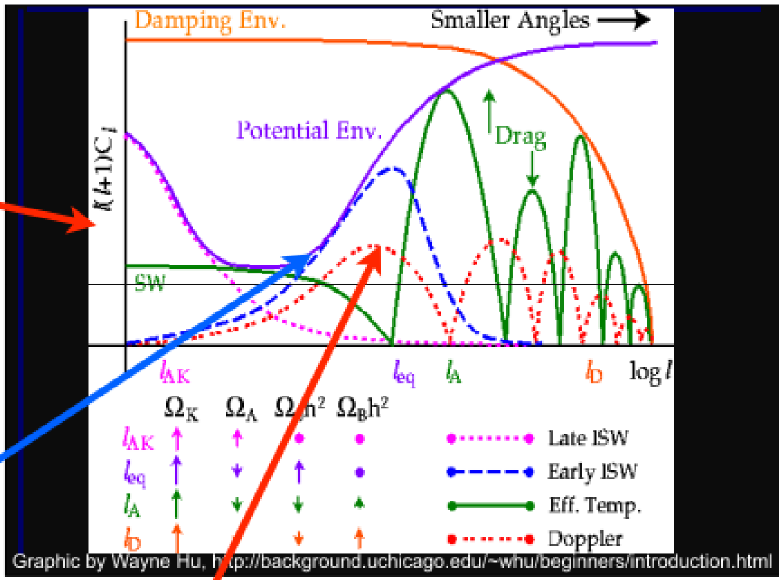
There is also a further reduction in amplitude of all the peaks owing to the fact that the surface of last scattering wasn't a single event but a slightly prolonged process.



Full spectrum with features

- Sachs-Wolfe plateau tells us the magnitude of the fluctuations in the initial scale-invariant anisotropy
- Location of first peak tells us timing of matter-radiation equality
- Ratio of odd to even peaks in the acoustic oscillations tells us the baryon density of the universe, since this effect is due to baryons
- The relative positions of the acoustic peaks tell us the geometry of the universe (using $k = \frac{\pi}{s_{rec}}$), or if you assume a flat universe, they determine H0

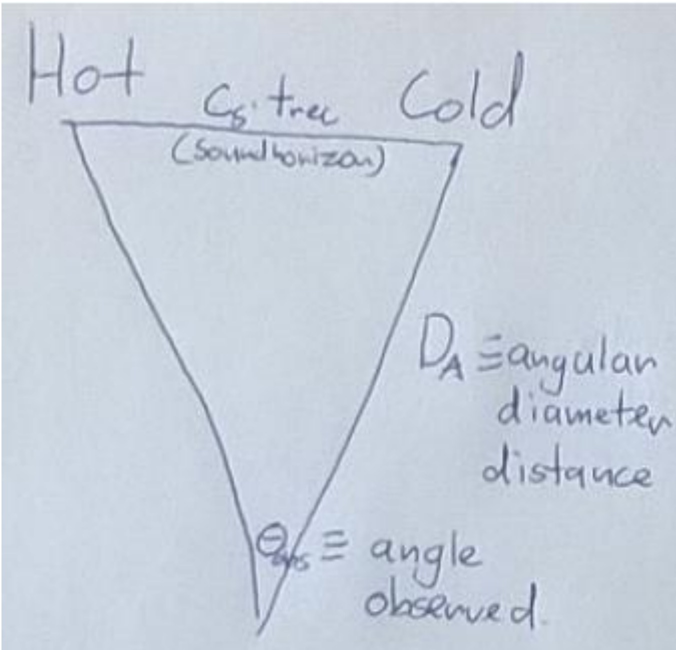




It's possible to break-down the power spectrum in a detailed manner

Also an early ISW effect due to the effect of photons climbing out of overdensities immediately after recombination

Doppler effect due to the expected motions of material in between modes at recombination epoch

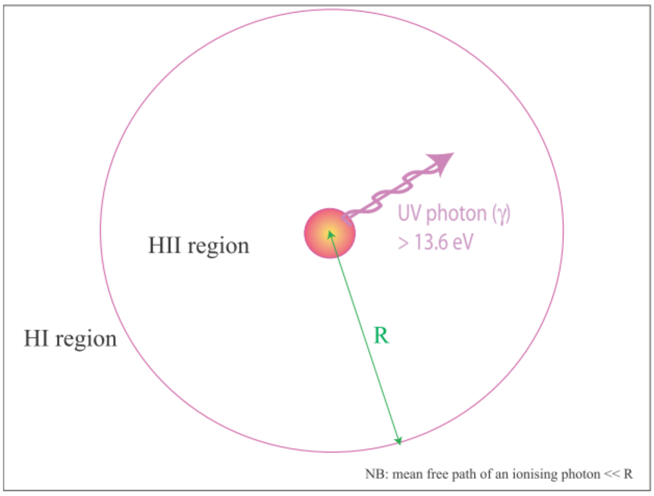


Reionisation

Basic Analytic Model

Reionisation is the process whereby the light emitted from the first galaxies includes ionising photons with energy sufficient (E>13.6 eV) to reionise the neutral hydrogen left in the intergalactic medium (IGM) following recombination. The volume of the reionised region will be determined by the number of photons emitted N_γ and the density of the HI in the IGM:

$$V_p = \frac{N_\gamma}{n_H}$$



We need to also account for recombinations. The recombination rate per unit volume will be given by:

$$r = n_H^2 \alpha_B$$

Where $\alpha_B = 2.6 \times 10^{-13} \mathrm{cm^3 s^{-1}}$.

The reionised HII region reaches equilibrium when recombinations balance ionisations:

$$\frac{dN_\gamma}{dt} = \alpha_B n_H^2 V_p$$

The corresponding radius R is called the Stromgren radius.

We also need to incorporate the fact that the IGM is expanding, thereby reducing n_H, and also the fact that the IGM is inhomogenous so we need to consider variations in $n_H = n_H^0 a^{-3}$:

$$\frac{d}{dt}(n_H V_p) + \frac{dN_\gamma}{dt} = \alpha_B \langle n_H^2 \rangle V_p$$

$$n_H \frac{dV_p}{dt} + V_p \frac{dn_H}{dt} = \frac{dN_\gamma}{dt} - \alpha_B \langle n_H^2 \rangle V_p$$

$$n_H \frac{dV_p}{dt} + n_H^0 V_p \frac{da^{-3}}{dt} = \frac{dN_\gamma}{dt} - \alpha_B \langle n_H^2 \rangle V_p$$

$$n_H \frac{dV_p}{dt} - 3 n_H^0 V_p \frac{\dot{a}}{a^4} = \frac{dN_\gamma}{dt} - \alpha_B \langle n_H^2 \rangle V_p$$

$$n_H \left(\frac{dV_p}{dt} - 3 V_p H \right) = \frac{dN_\gamma}{dt} - \alpha_B \langle n_H^2 \rangle V_p$$

We can define a clumping factor that specifies how clumped the matter distribution is:

$$C = \frac{\langle n_H^2 \rangle}{\langle n_H \rangle^2}$$

Using this and the definition $V = V_p a^{-3}$ we can rewrite our equation as:

$$\frac{dV}{dt} = \frac{1}{\langle n_H^0 \rangle} \frac{dN_\gamma}{dt} - \alpha_B C a^{-3} \langle n_H^0 \rangle V_p$$

The solution to this equation is given by:

$$V(t) = \int_{t_i}^{t_f} \frac{1}{\langle n_H^2 \rangle} \frac{dN_\gamma}{dt} \exp[F(t,t')]\, dt$$
$$F(t,t') = -\alpha_B \langle n_H^0 \rangle \int_t^{t'} \frac{C(t)}{a^3(t)} dt$$

For a universe with constant C and containing only matter this latter equation becomes:

$$F(t,t') = -\frac{2}{3}\frac{\alpha_B}{\sqrt{\Omega_m}}\frac{\langle n_H^0 \rangle}{H_0} C \left[a^{-\frac{3}{2}}(t') - a^{\frac{3}{2}}(t)\right]$$

The number of emitted (into the IGM) photons per baryon N_{ion} can be written:

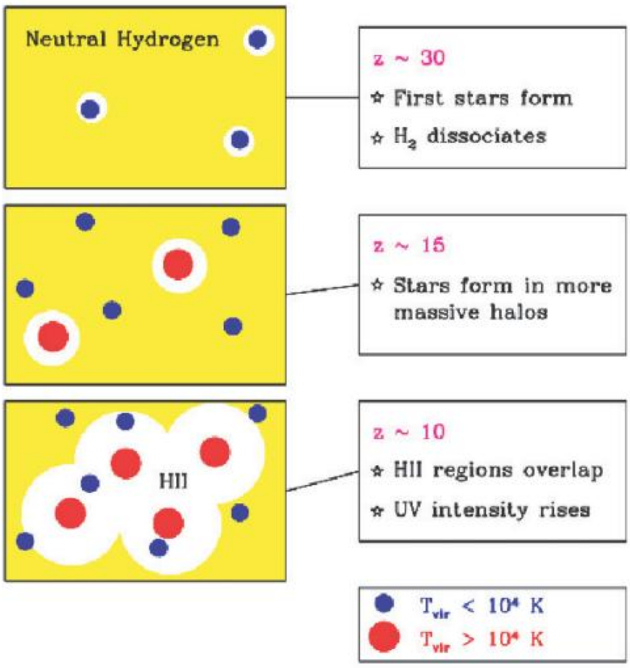
$$N_{ion} = f_* f_{esc} N_\gamma$$

Where f_* is the fraction of baryons (which are in the galaxy) that form stars, and f_{esc} is the fraction of ionising photons that leave the galaxy. Both are around 10%.

The radius reionised by a single galaxy is given by:

$$R_{\max} = \left(\frac{3}{4\pi} V\right)^{1/3} = 675 \text{ kpc} \left(\frac{N_{ion}}{40} \frac{M}{10^9 M_{sol}}\right)^{1/3}$$

This is considerably larger than the virial radius of around 30 kpc, indicating that a galaxy can ionise much more than its own volume.



Calculating the Filling Factor

We can try to calculate how much of the universe has been reionised by computing:

$$F = \frac{N_{bubbles} \times V_{bubble}}{V_{IGM}}$$

This definition, however, can result in $F > 1$, so instead we define the porosity Q:

$$Q = 1 - e^{-F}$$

Summing over all the individual bubbles and dividing by the total IGM volume we have:

$$\frac{1}{V_{tot}}\sum_i \frac{dV_i}{dt} = \frac{1}{V_{tot}}\sum_i \left[\frac{1}{\langle n_H^0\rangle}\frac{dN_\gamma}{dt} - \alpha_B C a^{-3}\langle n_H^0\rangle V_i\right]$$

Assuming the same scale factor for all i and a constant C we can write:

$$\frac{d}{dt}\sum_i \frac{V_i}{V_{tot}} = \frac{1}{\langle n_H^0\rangle}\frac{d}{dt}\sum_i \frac{N_\gamma^i}{V_{tot}} - \frac{\alpha_B C}{a^3}\langle n_H^0\rangle \sum_i \frac{V_i}{V_{tot}}$$

$$\frac{d}{dt}F = \frac{1}{\langle n_H^0\rangle}\frac{dn_\gamma^0}{dt} - \frac{\alpha_B C}{a^3}\langle n_H^0\rangle F$$

Where n_γ^0 is the commoving photon density. We can write this in terms of baryon density:

$$\langle n_H^0\rangle = 0.76\langle n_b^0\rangle$$

The photon to baryon ratio can be written in terms of:

$$\frac{n_\gamma^0}{\langle n_H^0\rangle} = \frac{n_\gamma^0}{0.76\langle n_b^0\rangle} = \frac{N_{ion}F_{col}}{0.76}$$

Yielding the equation:

$$\frac{dF}{dt} = \frac{N_{ion}}{0.76}\frac{dF_{col}}{dt} - \frac{\alpha_B C}{a^3}\langle n_H^0\rangle F$$

Which has the solution:

$$F(t) = \int_{t_i}^{t_f} \frac{N_{ion}}{0.76}\frac{dF_{col}}{dt}\exp[F_*(t,t')]\,dt$$

$$F_*(t,t') = -\alpha_B\langle n_H^0\rangle \int_t^{t'} \frac{C(t)}{a^3(t)}dt$$

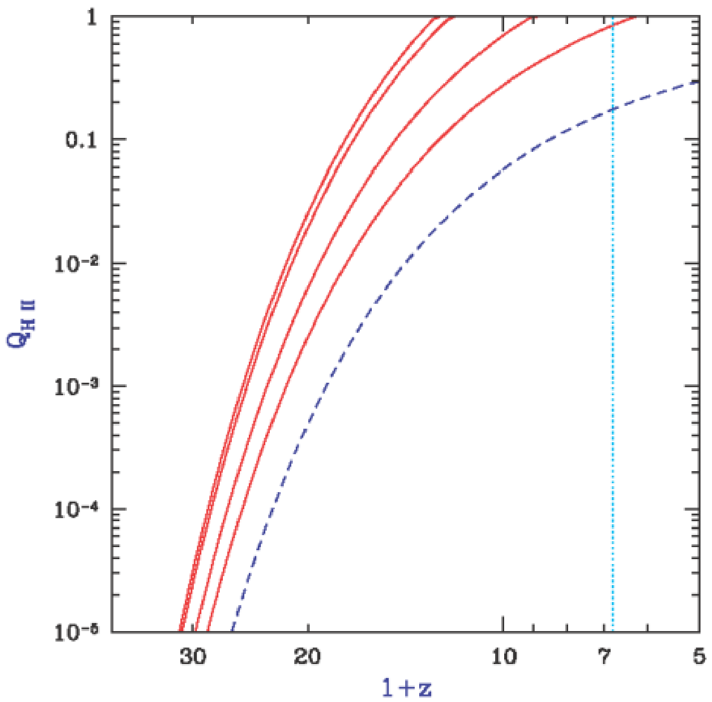


Figure 34: From Barkana and Loeb, Fig 22. [1] "Semi-analytic calculation of the reionization of the IGM (for $N_{ion} = 40$), showing the redshift evolution of the filling factor Q_{HII}. Solid curves show Q_{HII} for a clumping factor C = 0 (no recombinations), C = 1, C = 10, and C = 30, in order from left to right. The dashed curve shows the collapse fraction F_{col}, and the vertical dotted line shows the z = 5.8 observational lower limit (Fan et al. 2000) on the reionization redshift."

Observations of Reionisation

We want to determine the neutral hydrogen content of the IGM by calculating the optical depth $\tau(\lambda_{obs})$ for Ly-α photons. This is given by the equation:

$$\tau = \int_0^{z_{source}} c\frac{dt}{dz}\langle n_H^0\rangle(1+z)^3\sigma_\alpha\big(\nu_{obs}(1+z)\big)\,dz$$

Where $\langle n_H^0\rangle(1+z)^3$ is the physical density of hydrogen, $\sigma_\alpha\big(\nu_{obs}(1+z)\big)$ is the scattering cross-section, and $c\frac{dt}{dz}$ is the path length. One observational method found the value:

$$\tau = 6.45\times10^5\times x_{HI}\left(\frac{1+z_s}{10}\right)^{3/2}$$

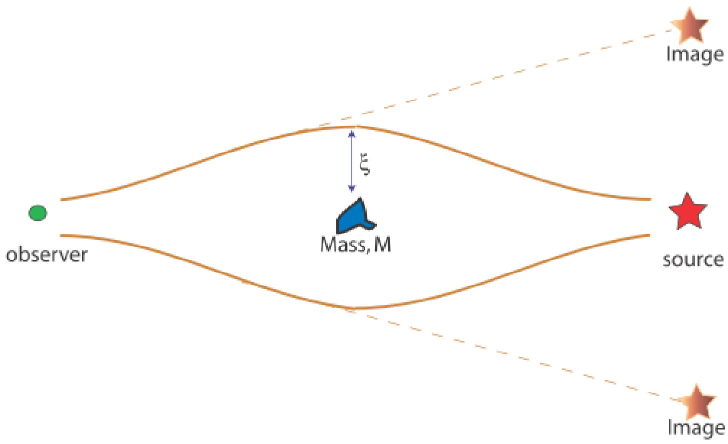
Studying the Thompson scattering of CMB photons (causing dampening of the acoustic peak) also provides an additional observational lens upon reionisation.

Based on such measurements we expect reionisation to have occurred during a period of around $6 < z_{re} < 10$.

Gravitational Lensing

Basic Principles

Gravitational lensing is caused by the bending of spacetime by mass. It results in lensed objects appearing larger and/or brighter than they otherwise would.



$$\mu = \frac{\text{image area}}{\text{source area}}$$

The general thin-screen approximation solution for the bending angle is given by:

$$\alpha(\xi) = \frac{4\pi G}{c^2} \int \Sigma(x) \frac{\xi - x}{|\xi - x|^2} d^2x$$

For a spherical mass distribution of any sort this simplifies to:

$$\alpha(\xi) = \frac{4GM(<\xi)}{c^2\xi}$$

Where $\Sigma(x, y)$ is the surface density over the 'face' of the object visible to us (i.e. from our line-of-sight perspective), with the formula:

$$\Sigma(\xi) = \int_{-\infty}^{\infty} \rho \, dz$$

The mass can be calculated from the surface density by:

$$M(<\xi) = 2\pi \int_0^{\xi} \xi\Sigma(\xi) \, d\xi$$

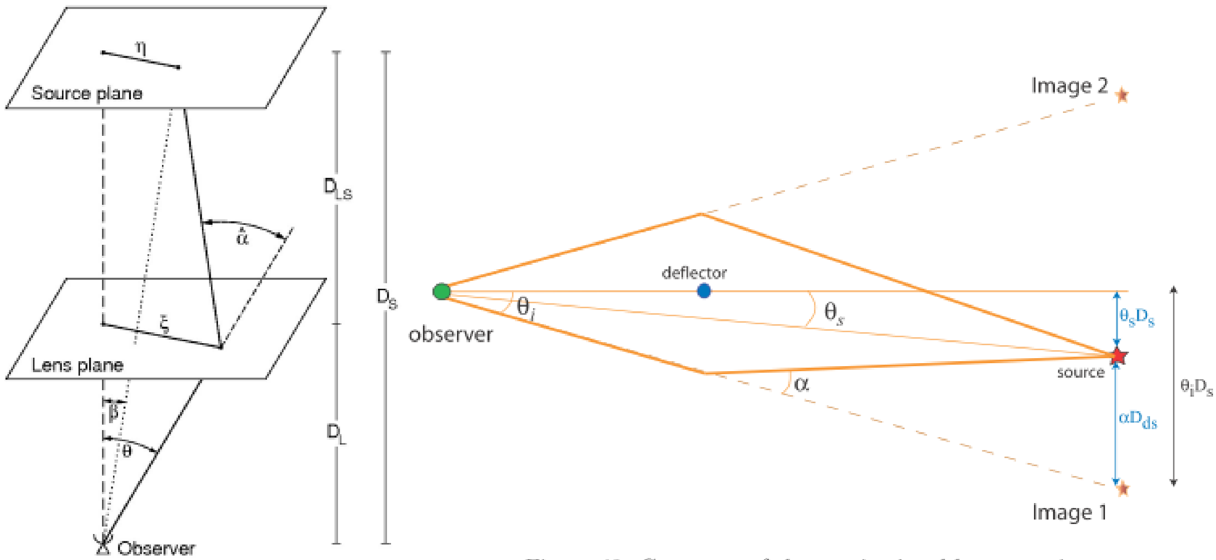


Figure 40: Geometry of the gravitational lens equation

Given the geometry as shown above, and defining D_d as the distance to the deflector/lens and D_s as the distance to the source, we can write down a lens equation as follows:

$$D_s\theta_i = D_s\theta_s + D_{ds}\alpha$$
$$\theta_s = \theta_i - \frac{D_{ds}}{D_s}\alpha$$

The bending angle does not depend on the wavelength of light, however differential magnification across a non-uniform source can give different magnifications to different wavelengths. Surface brightness is conserved, but a source may change shape and size. Polarisation is unaffected in the weak field case. The maximum possible magnification (limited by diffraction effects) is:

$$\mu_{max} = \frac{r_s}{\lambda}$$

For a background source of radius r to be strongly lensed by a foreground object, the angular size of the source must be smaller than the Einstein radius. For a foreground object to be a strong lens, its surface density must exceed the critical density for some positive Einstein ring radius.

Strong lensing condition:

$$r_s < \theta_{ER}D_s$$

Einstein Radius

The Einstein radius is the radius of the Einstein Ring, which maps the critical curve inside which the region has an average surface density of:

$$\Sigma_{crit} = \frac{c^2}{4\pi G}\frac{D_s}{D_d D_{ds}}$$

The Einstein radius is defined for when $\theta_s = 0$:

$$\alpha = \frac{D_s}{D_{ds}}\theta_i$$

The radius itself is defined in terms of an angular distance and for a spherical mass is given by the formula:

$$\alpha = \frac{4GM(<\xi)}{c^2\xi}$$
$$\theta_{ER}\frac{D_s}{D_{ds}} = \frac{4GM(<\xi)}{c^2\xi}$$
$$\theta_{ER}\frac{D_s}{D_{ds}} = \frac{4GM(<\theta_{ER}D_d)}{c^2\theta_{ER}D_d}$$
$$\theta_{ER}^2 = \frac{D_{ds}}{D_sD_d}\frac{4GM(<\theta_{ER}D_d)}{c^2}$$
$$\theta_{ER} = \sqrt{\frac{D_{ds}}{D_sD_d}\frac{4GM(<\theta_{ER}D_d)}{c^2}}$$
$$\theta_{ER} \approx 2'' \left(\frac{M}{10^{12}\,M_{solar}}\right)^{1/2}\left(\frac{D_sD_d/D_{ds}}{0.3\text{ Gpc}}\right)^{-1/2}$$

If the surface density within the Einstein radius exceeds the critical value Σ_{crit}, then multiple images of the object will be observed at the Einstein radius.

Note that D_sD_d/D_{ds} is maximised for $D_s = D_d$.

Point Mass

For a point mass the mass is constant, and thus the surface density is:

$$\Sigma(x) = \delta_{dir}(x)M$$

Rendering the deflection angle simply:

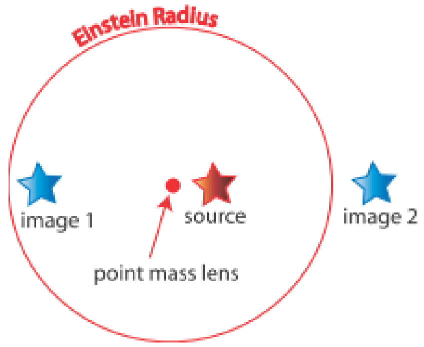
$$\alpha(\xi) = \frac{4\pi GM}{c^2\xi}$$

The lens equation therefore becomes:

$$\theta_s = \theta_i - \frac{D_{ds}}{D_s}\alpha$$
$$= \theta_i - \frac{D_{ds}}{D_s}\frac{4\pi GM}{c^2\xi}$$
$$= \theta_i - \frac{D_{ds}}{D_s}\frac{4\pi GM}{c^2\theta_iD_d}$$
$$= \theta_i - \frac{1}{\theta_i}\left(\frac{D_{ds}}{D_sD_d}\frac{4\pi GM}{c^2}\right)$$
$$\theta_s = \theta_i - \frac{\theta_{ER}^2}{\theta_i}$$

This has two solutions, one inside and one outside the Einstein radius:

$$\theta_i = \frac{1}{2}\left(\theta_s \pm \sqrt{\theta_s^2 + 4\theta_{ER}^2}\right)$$



The total magnification becomes:

$$\mu = \frac{A_{im}}{A_{sor}}$$
$$= \frac{\theta_i}{\theta_s}\frac{d\theta_i}{d\theta_s}$$
$$\mu = \left[1 - \left(\frac{\theta_{ER}}{\theta_i}\right)^4\right]^{-1}$$

Note that at $\theta_i = \theta_{ER}$ we get $\mu = \infty$, but only for an infinitesimally small source. A source at the Einstein radius we have a magnification of:

$$\mu = \left[1 - \left(\frac{\theta_{ER}}{\frac{1}{2}\left(\theta_{ER} \pm \sqrt{\theta_{ER}^2 + 4\theta_{ER}^2}\right)}\right)^4\right]^{-1}$$
$$= \left[1 - \left(\frac{2\theta_{ER}}{\theta_{ER} \pm \sqrt{5}\theta_{ER}}\right)^4\right]^{-1}$$
$$= 0.17 \; or \; 1.17$$
$$\mu_{tot} = |\mu_1| + |\mu_2|$$
$$= 1.34$$

Negative values of µ imply reversed parity. Images within the Einstein ring have negative parity while those outside have positive parity.

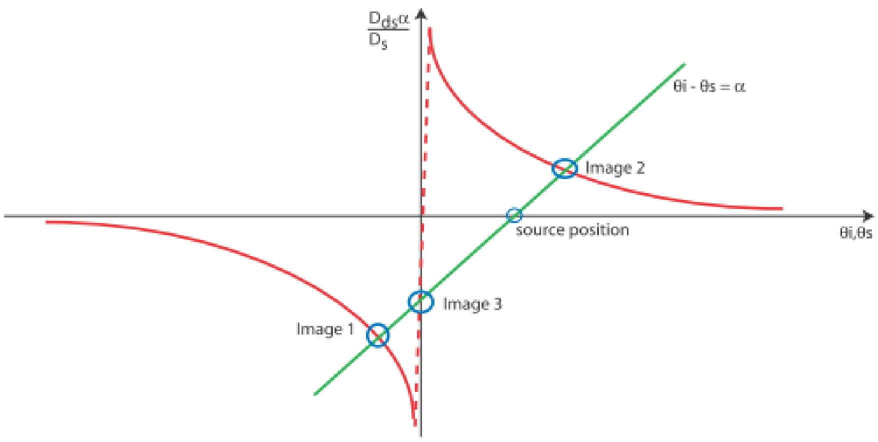


Figure 47: Graphical representation of the lens equation for a point mass. Note that for all source positions (x intercept) there are 2 images, plus a 3rd image with $\mu = 0$ since $d\theta_i/d\theta_s = 0$

Singular Isothermal Sphere

A gas of stars (i.e. a galaxy) with constant temperature (and hence constant velocity) can be modelled as an isothermal sphere, which has a density given by:

$$\rho = \frac{\sigma^2}{2\pi G}\frac{1}{\xi^2 - z^2}$$

The surface density for such a mass distribution is:

$$\begin{aligned}\Sigma(\xi) &= \int_{-\infty}^{\infty} \rho \, dz \\ &= \frac{\sigma^2}{2\pi G}\int_{-\infty}^{\infty}\frac{1}{\xi^2 - z^2} dz \\ \Sigma(\xi) &= \frac{\sigma^2}{2G}\frac{1}{\xi}\end{aligned}$$

Using this we can find the mass distribution:

$$\begin{aligned}M(<\xi) &= 2\pi \int_0^{\xi} \xi \Sigma(\xi) \, d\xi \\ &= 2\pi \int_0^{\xi} \xi \frac{\sigma^2}{2G}\frac{1}{\xi} d\xi \\ &= \frac{\pi\sigma^2}{G}\int_0^{\xi} d\xi \\ M(<\xi) &= \frac{\pi\sigma^2}{G}\xi\end{aligned}$$

The bend angle is then (using the formula for circular mass):

$$\begin{aligned}\alpha(\xi) &= \frac{4GM(<\xi)}{c^2\xi} \\ \alpha(\xi) &= \frac{4\pi\sigma^2}{c^2}\end{aligned}$$

The Einstein radius is found by:

$$\begin{aligned}\theta_{ER} &= \frac{D_{ds}}{D_s}\alpha \\ \theta_{ER} &= \frac{D_{ds}}{D_s}\frac{4\pi\sigma^2}{c^2}\end{aligned}$$

The lens equation then becomes:

$$\begin{aligned}\theta_s &= \theta_i - \frac{D_{ds}}{D_s}\alpha \\ &= \theta_i - \frac{D_{ds}}{D_s}\frac{4\pi\sigma^2}{c^2} \\ \theta_s &= \theta_i - \theta_{ER}\end{aligned}$$

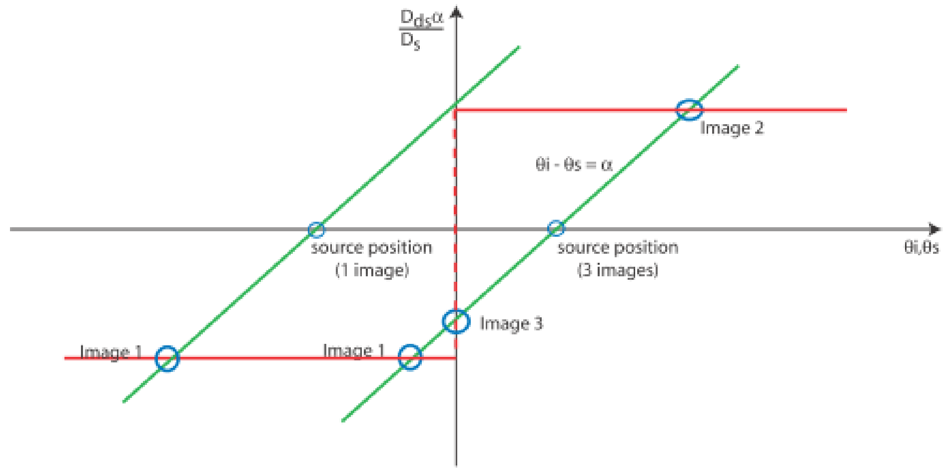


Figure 48: Graphical representation of the lens equation for a SIS. Note that when the source is outside 1 ER there is only 1 image, while if it it inside there will be 3 (one of which has zero magnification).

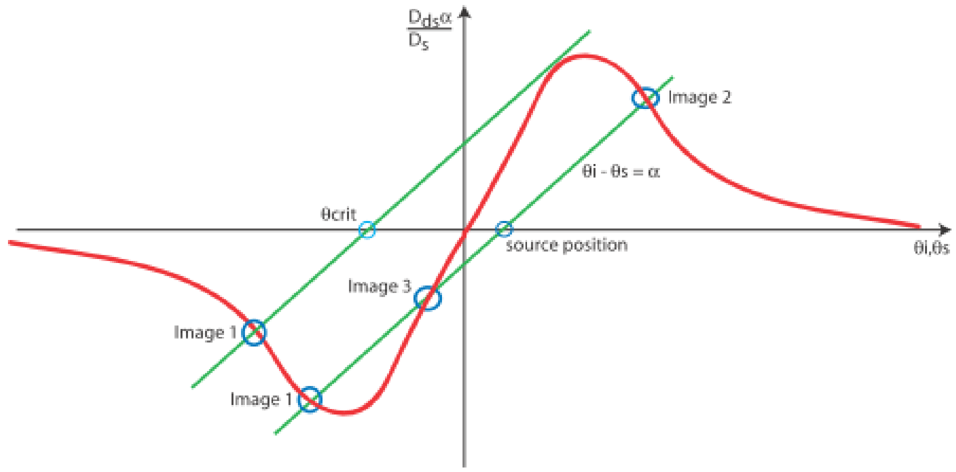


Figure 49: Graphical representation of the lens equation for a general circular lens. A source position outside θ_{crit} results in a single image, while within θ_{crit} there are 3 images.

Microlensing in the Galaxy

Stars in the galaxy are viewed through our own galaxy, and therefore through a large collection of point masses. If a background source (e.g. a galaxy) is moving relative to a point source lens (e.g. a star in our galaxy) we will see a variable magnification, with μ_{tot} greater than 1.34 then when it is inside the Einstein Radius.

The time scale of this varying magnification is given by:

$$t_d = \frac{\xi_{ER}}{v}$$
$$= \frac{\theta_{ER} D_d}{v}$$
$$t_d \approx 0.214 \text{ yrs} \left(\frac{M}{M_{solar}}\right)^{\frac{1}{2}} \left(\frac{D_d}{10 \text{ kpc}}\right) \left(\frac{D_{ds}}{D_s}\right)^{\frac{1}{2}} \left(\frac{v}{200 \text{ km/s}}\right)^{-1}$$

During this interval of time when the source is in the Einstein radius, we have $\mu_{tot} < 1.34$ and the image separation given by about $2\theta_{ER}$, which is about:

$$sep \approx 0.9 \text{ milli arcsec } \left(\frac{M}{M_{solar}}\right)^{\frac{1}{2}} \left(\frac{D_d}{10 \text{ kpc}}\right)^{-\frac{1}{2}}$$

To determine how many lenses are in the galaxy/halo, we can count the fraction of the sky covered by Einstein Radii, which occur when sources are magnified by $\mu > 1.34$. The optical depth is:

$$\tau = \int_0^{D_s} n_{lenses} \times a_{lensed} \times a_{to\ lens}\, dD_d$$

$$\tau = \int_0^{D_s} n(D_d)\theta_{ER}^2 \pi D_d^2\, dD_d$$

Using $n = \frac{\rho}{M}$ and $\theta_{ER}^2 = \frac{D_{ds}}{D_d D_s}\frac{4GM}{c^2}$ we write:

$$\begin{aligned} \tau &= \int_0^{D_s} \frac{\rho}{M}\frac{D_{ds}}{D_d D_s}\frac{4\pi GM}{c^2}\pi D_d^2\, dD_d \\ &= \frac{4\pi G}{c^2}\int_0^{D_s} \rho \frac{(D_s - D_d)D_d}{D_s}\, dD_d \\ \tau &= \frac{4\pi G}{c^2}\int_0^{D_s} \rho \left(1 - \frac{D_d^2}{D_s}\right) dD_d \end{aligned}$$

Defining $x = D_d/D_s$ we can write:

$$\tau = \frac{4\pi G}{c^2} D_s^2 \int_0^1 \rho(1 - x)x\, dx$$

If the density is roughly constant this becomes:

$$\tau = \frac{2\pi}{3}\frac{G\rho}{c^2} D_s^2$$

Observationally we find that $\tau \sim 10^{-7}$ towards the large magellanic cloud, and $\tau \sim 10^{-6}$ towards the galactic center. This implies that only about 20% of dark matter could be unseen MACHO objects.

A single measure can be derived for the probability of lensing a background source by a galaxy. It has the form:

$$\tau = \frac{c}{H_0}\int\int_0^{z_s} \frac{dn(z)}{dM}\theta_{ER}^2 \pi (1+z)^{\frac{1}{2}} D_d^2\, dz\, dM$$

Any survey of the sky only sees down to a minimum flux level. However lensed sources are magnified, and so can appear in the sample in cases where they would be intrinsically too faint to be in the sample in the absence of magnification. This is called magnification bias. For quasars, for example, the probability of being magnified above the threshold is around 0.008.