

Part 1: Philosophy and Semantics

Reference

Names stand for *objects*

Statements stand for *truth values*

Predicates stand for *extensions*

The extension of a predicate – a truth-valued function – is the set of tuples of values that, used as arguments, satisfy the predicate. For example, the predicate: "d2 is the weekday following d1" has the extension:

{(Monday, Sunday), (Tuesday, Monday), (Wednesday, Tuesday), (Thursday, Wednesday), (Friday, Thursday), (Saturday, Friday), (Sunday, Saturday)}

It seems, however, that extensions are not enough for meaning:

What is the extension of
'Female Prime Minister of Australia before July 2010'?

What is the extension of
'is identical to Julia Gillard'?

They're the same extension: {Julia Gillard}.

Do these predicates *mean* the same thing?

There also seem to be problems with identity statements:

Do we all *know* that $e^{i\pi} + 1 = 0$

simply because we know that $0 = 0$,

and because $e^{i\pi} + 1$ really is 0?

Frege's distinction rejects a view put forward by John Stuart Mill, according to which a proper name has no meaning above and beyond the object to which it refers (its referent or reference). Frege's central objection to the view that a name's meaning is no more than its referent is that, if a and b are names of the same object, then the identity statement $a = b$ must mean the same as $a = a$. Yet clearly the first can convey information in a way that the second cannot; that Samuel Clemens is Samuel Clemens is just trivial, but that Samuel Clemens is Mark Twain is interesting.

Frege thus introduced the distinction between sense and reference. He postulated that, in addition to a reference (Bedeutung), a proper name possesses what he calls a sense (Sinn), some aspect of the way its reference is thought of that can differ, even between two names that refer to the same object. The reference is the object that the expression refers to. The sense is the "cognitive significance" or "mode of presentation" of the referent. Two names that name the same thing might have different senses: two predicates with the same extension might have different senses.

Classical Logic is about as far as you can go with reference (extension) alone. Sense-sensitive expressions (modals, conditionals, etc.) require that we go beyond truth values and extensions.

Compositionality

The value of a complex statement depends on the value of its components. Proponents of compositionality typically emphasize the productivity and systematicity of our linguistic understanding. We can understand a large—perhaps infinitely large—collection of complex expressions the first time we encounter them, and if we understand some complex expressions we tend to understand others that can be obtained by recombining their constituents. Opponents of compositionality typically point to cases when meanings of larger expressions seem to depend on the intentions of the speaker, on the linguistic environment, or on the setting in which the utterance takes place without their parts displaying a similar dependence.

A connective is used truth functionally to form a sentence from components if and only if that sentence's truth value depends only on the truth value of the components. Otherwise, it is used non-truth functionally.

The basic idea is this: suppose we have a statement connective, call it +, and suppose we have any two statements, call them S1 and S2. Then we can form a compound, which is denoted S1+S2. Now, to say that the connective '+' is truth-functional is to say that if we know the truth values of S1 and S2 individually, then we automatically know, or can compute, the truth value of S1+S2. On the other hand, to say that the connective '+' is not truth-functional is to say that merely knowing the truth values of S1 and S2 does not automatically tell us the truth value of S1+S2.

An example of a non-truth functional connective is the subjunctive conditional, the value of which depends on facts about the world aside from the truth value of the statements involved. For example consider the two conditionals:

- If I lived in L.A., then I would live in California
- If I lived in N.Y.C., then I would live in California

Three propositions are involved here:

- I live in L.A. (Los Angeles)
- I live in N.Y.C. (New York City)
- I live in California

All of these are false, yet the subjunctive conditional would say the first statement is true but the second is false. Hence this type of conditional is not truth-functional, because to determine its truth value we needed to know more than merely the truth value of its constituent parts. Statements involving temporal sequence or causation are often not truth-functional.

Possibility and Necessity

‘ $\Box A$ ’ means ‘A is necessary.’

‘ $\Diamond A$ ’ means ‘A is possible.’

- ▶ **PHYSICAL NECESSITY:** necessarily there’s no travelling faster than light
- ▶ **LOGICAL NECESSITY:** it is necessary that if A and B, then $A \wedge B$.
- ▶ **MORAL NECESSITY:** it is necessary to keep your promises.

Logic	Symbols	Expressions Symbolized
Modal Logic	\Box	It is necessary that ..
	\Diamond	It is possible that ...
Deontic Logic	O	It is obligatory that ...
	P	It is permitted that ...
	F	It is forbidden that ...
Temporal Logic	G	It will always be the case that ...
	F	It will be the case that ...
	H	It has always been the case that ...
	P	It was the case that ...
Doxastic Logic	Bx	x believes that ...

What are Possible Worlds?

- Modal realism: Possible worlds are other things just like the actual world, albeit spatially and causally disconnected from the actual world
 - Objections: 'incredulous stare', inflated cosmology, distinction between mathematical and physical reality
- Modal actualism: Possible worlds exist but are not like the actual world — they are abstract things like numbers, properties, etc
 - Objections: what type of abstract objects exactly?
- Meinongianism: Possible worlds don’t exist, but we can still talk about them, just as we can talk about other things which don’t exist - e.g. we can say that unicorns have horns, even though unicorns don't exist. Hence, despite not existing, they feature in explanations
 - How to know about fictional objects, why useful if possible worlds don't exist?
- Representationalism: There are no other possible worlds. Possible world talk represents this world. In just the same way that we can represent how the world is, we can also represent how the world could have been

Arthur Prior: world-talk isn’t actually needed for explaining modal claims. We understand ‘truth in a possible world’ because we understand ‘necessarily’; not vice versa.

Part 2: Necessity and Possibility

Metaphysical Necessity

$\Box A$ means “it is absolutely necessary that A: there is *no* way, no matter how odd or crazy, according to which that A have is false.”

$\Diamond A$ means “there is some way, perhaps very odd or crazy, that things could have been such that A were true.”

The set W of worlds is *all* of the different ways that the world could have been.

The relation R is *universal*.

The logic of Leibniz’s Metaphysical Necessity is S_5 .

Physical Necessity

$\Box A$ means “it follows from the laws of physics that A.”

$\Diamond A$ means “the laws of physics allow that A.”

The set W is still all of the different ways that the world could have been (even ones with different physical laws than ours).

The relation R is this: wRu if and only if the laws of physics in w are satisfied in u .

Deontic Logic

$\Box A$ means “it is obligatory that A.”

$\Diamond A$ means “it is allowed that A.”

The set W is still all of the different ways that the world could have been (even ones with different norms of obligation than ours).

The relation R is this: wRu if and only if the norms in w are satisfied in u .

Epistemic Logic

Some people think you can model the logic of belief or knowledge using modal logic. $\Box_\beta A$ means person β believes that A .

Take the worlds to be the collection of all different ways that the world could be.

We'll say that $xR_\beta y$ if and only if whenever what β takes to be true in world x is *actually* true in world y . So y is a world which is (at least) how β (in x) believes it to be.

Temporal Logic

$\Box A$ means "always (in the future) A ."

$\Diamond A$ means "at some (future) time A ."

The set W is the set of all *times*.

The relation R is this: wRu iff u is later than w . We will typically write this relation as $w < u$

- End of change: $\vdash \Diamond p \supset \Diamond(q \supset \Box q)$
- End of time: $\vdash \Box p \vee \Diamond \Box p$ (true as nothing is possible but everything is necessary)
- Time doesn't diverge: $\vdash \Diamond \Box p \supset \sim \Diamond \Box \sim p$
- Time is circular: $\Box A, 1 \vdash A, 1$

Actual World

In the collection of worlds, we can single out one as the actual world. Call it $w@$.

Only Two Worlds

The following formula is true in every S5 model with only two worlds, but can be falsified in models with more than two worlds:

$$(\Diamond p \wedge \Diamond q \wedge \Diamond r) \supset (\Diamond(p \wedge q) \vee \Diamond(p \wedge r) \vee \Diamond(q \wedge r))$$

In any S5 model with only two worlds, if p , q and r are all possible then we must have two of them true in the same world. In a model with more than two worlds we can make this formula false by having p , q and r all true in the model, but at distinct worlds.

Part 3: Conditionals and Two-Dimensions

Contingent A Priori

- ▶ NECESSARY: true in every possible world.
- ▶ ANALYTIC: true in virtue of the meanings of the words/concepts.
- ▶ A PRIORI: can be known independently of external evidence.

In two-dimensional modal logic, there seem to be some claims that are contingent (not true in every possible world), but can be known a priori (independently of external evidence). A simple example is $p \supset @p$, which is always true in the actual world (so can be known a priori), but is not true in every possible world (it may be false in non-actual worlds).

This notion is even broader than the notion of actual worlds. Davies and Humberstone (1980) provide an analysis of the contingent a priori which relies on their analysis of actuality. Actuality, in these models, is a kind of *indexicality*. Relative to the context of use, the actual world is just the world in which the context of use is found. In the same way, other indexicals such as 'I', 'now' and 'here' pick out the speaker, the time and the location in the context of use.

Davies and Humberstone show how even putative synthetic and contingent a priori claims, using definitions such as "Julius invented the zip, if anyone did" (which is a priori but contingent if "Julius" is introduced as "the inventor of the zip, if the zip had an inventor"), can be understood as involving the notion of actuality. "Julius" is not shorthand for "the inventor of the zip, if the zip had an inventor" (which would result in a necessary truth "the inventor of the zip, if the zip had an inventor, invented the zip") but as "the person who actually invented the zip, if the zip had an inventor."

So, are there any contingent a priori claims that don't involve actuality like these? One example of an a priori statement that does not seem to use any indexicals, is: "There is at least one thinker". This is contingent, because had life not evolved here or elsewhere, there would have been no thinkers.

However, if I entertain this proposition, I don't need to collect any evidence for it to know that it is true. There is no way to think the "There is at least one thinker" is to notice that I am thinking in the very act of considering the statement (so I conclude "I am a thinker"), and to infer from this that there is at least one thinker. Although there is an indexical in "I am a believer", there is no indexical in the weaker (but still contingent) "There is at least one believer".

Strict Implication

The following two arguments are valid according to the material conditional:

- Positive Paradox, $A \vdash B \supset A$: "I'm alive. Therefore, if I'm dead then I'm alive"
- Negative Paradox, $\sim A \vdash A \supset B$: "WWII did not end in 1942. Therefore, if WWII did end in 1942 then gold is an acid"
- Conjunction Paradox, $\vdash A \supset B \vee B \supset A$: "If it's Saturday it's raining, or if it's raining then it's Saturday"

These and other failings of the material conditional have led many to propose alternatives.

One reason modal logics were invented was to give an account of conditionality.

A new conditional: the *strict* conditional, is defined like this:

$$A \rightarrow B \text{ is } \Box(A \supset B)$$

Strict implication has advantages in modelling our ‘intuitive’ notion of conditionality because it avoids what we can call the positive and negative paradoxes:

- ▶ POSITIVE: $p \not\vdash q \rightarrow p$
- ▶ NEGATIVE: $p \not\vdash \neg p \rightarrow q$

These *are* valid when we replace \rightarrow by \supset .

However, the strict conditional does not address these other problems:

- Antecedent Strengthening, $A \supset B \vdash (A \wedge B) \supset B$: "If I have a cup of coffee, I feel good. So, if I have a cup of coffee with poison, I feel good"
- Transitivity, $A \supset B, B \supset C \vdash A \supset C$: "If an election is held on December 25, it will be held in December. If an election is held in December, it won't be held on December 25. So, if an election is held on December 25, it won't be held on December 25"
- Contraposition, $A \supset \sim B \vdash B \supset \sim A$: "If an election is held in December, it won't be held on December 25. So, if an election is held on December 25, it won't be held in December"

Similarity Conditionals

- ▶ We can think of $xR_A y$ as telling us that y is one of the worlds most similar to x which makes A true.
- ▶ On this account being ‘ceteris paribus the same as x but making A true’ means ‘being as similar as possible to x while making A true’.
- ▶ We can make this idea more precise: we can think of every world $w \in W$ as being surrounded by *similarity spheres* of worlds. That is we suppose that there are subsets S_i^w (for $0 \leq i \leq n$) of W such that:

$$w \in S_0^w \subseteq S_1^w \dots \supseteq S_n^w = W$$

Here the worlds in S_0^w are the worlds most like w , those in S_1^w are those where we have to make a small changes, etc.

One way to think of models for conditional logic where we think in terms of similarity spheres is to extend models for c^+ by the following conditions.

- (3) If $[A] \neq \emptyset$ then $f_A(w) \neq \emptyset$
- (4) If $f_A(w) \subseteq [B]$ and $f_B(w) \subseteq [A]$ then $f_A(w) = f_B(w)$
- (5) If $f_A(w) \cap [B] \neq \emptyset$ then $f_{A \wedge B}(w) \subseteq f_A(w)$

Call the system which extends c^+ by the following three conditions 's'.

S is a proper extension of c^+ , as is illustrated by the following arguments invalid in c^+ , but valid in s:

$$\Diamond A \models \neg(A > (B \wedge \neg B))$$

$$A > B, B > A \models (A > C) \supset (B > C)$$

$$\neg(A > \neg B) \models (A > C) \supset ((A \wedge B) > C)$$

One major limitation of similarity conditionals is the necessary vagueness of the notion of 'similarity', and how this can be applied across possible worlds.

Part 4: Many-Valued Logics

Liar Paradox

One way of understanding truth is to treat it as a predicate T

- T is a *predicate*.
- For each sentence A add to our language a *name* $\langle A \rangle$ for A.
- Tarski's *Convention T* is each sentence of this form: $T\langle A \rangle \equiv A$.
- $T\langle A \rangle$ should have the same truth value as A.

This analysis seems plausible, but runs into problems with the "Liar Sentence"

Consider the following sentence:

(λ) λ is not true.

Is λ true?

It seems that Convention T says that if it is, it isn't, and if it isn't it is.
And hence, that it is, and it isn't.

Monster-Barring Response

- Deny that such sentences can be formed, or at least that they make statements
- Banning self reference seems to be a very extreme response: what about "This sentence is in English"?

Reject Convention T

- But when it goes wrong, how does it go wrong?
- Do we sometimes accept A and reject $T(A)$, or vice versa? Both seem a bit weird

Trivialism

- You could bite the bullet hard and argue that the entire game of truth-talk is trivial
- Either everything is true (accept everything) or that nothing is (reject everything)
- We'd have to live without the idea of declarative talk as conveying information by classifying

Fixed Point Construction

The final option to resolve the liar paradox is to revise our theory of logic so that the argument from Convention T to triviality breaks down. Remember, the paradox is that:

$$T(\lambda) \equiv \sim T(\lambda)$$

This leads to contradiction and hence explosion in classic two-valued logic, but not in three-valued logic where we allow for gaps and/or gluts. In such logics, we can give $T(\lambda)$ and $\sim T(\lambda)$ the same value. Thus, admitting an intermediate value like i allows us to deal with the paradox.

Kripke was able to show rigorously that this process works using his fixed point construction. In order for the proof to work, it is necessary that the logic obey monotonicity:

MONOTONICITY FACT: If $v_1 \sqsubseteq v_2$ then for every *formula* A, if $v_1(A) = 0$ then $v_2(A) = 0$ and $v_1(A) = 1$ then $v_2(A) = 1$.

If more things become fixed (true or false) among the atoms, then more things become fixed in the rest of the language, and nothing becomes unfixed.

This means that settling an atomic proposition's truth value from i to 0 or 1 should never change the value of the overall proposition (under the relevant operator) from 0 to 1 or from 1 to 0, nor should it cause a 0 or 1 to revert to i . By examining the relevant truth tables, we see that this condition is satisfied for \vee, \wedge, \sim and for \supset in K3 and LP, but not for the \supset in L3 or RM3.

The idea is that every statement of the form $T(A), T(T(A))$, etc begins with the intermediate value. At each iteration, one layer of T is "peeled off", and assigned a value equal to the value of A .

This generalises. Given an interpretation v_i , define v_{i+1} by taking $v_{i+1}(T\langle A \rangle) = v_i(A)$, and v_{i+1} of every other atomic sentence is just the same as v_0 .

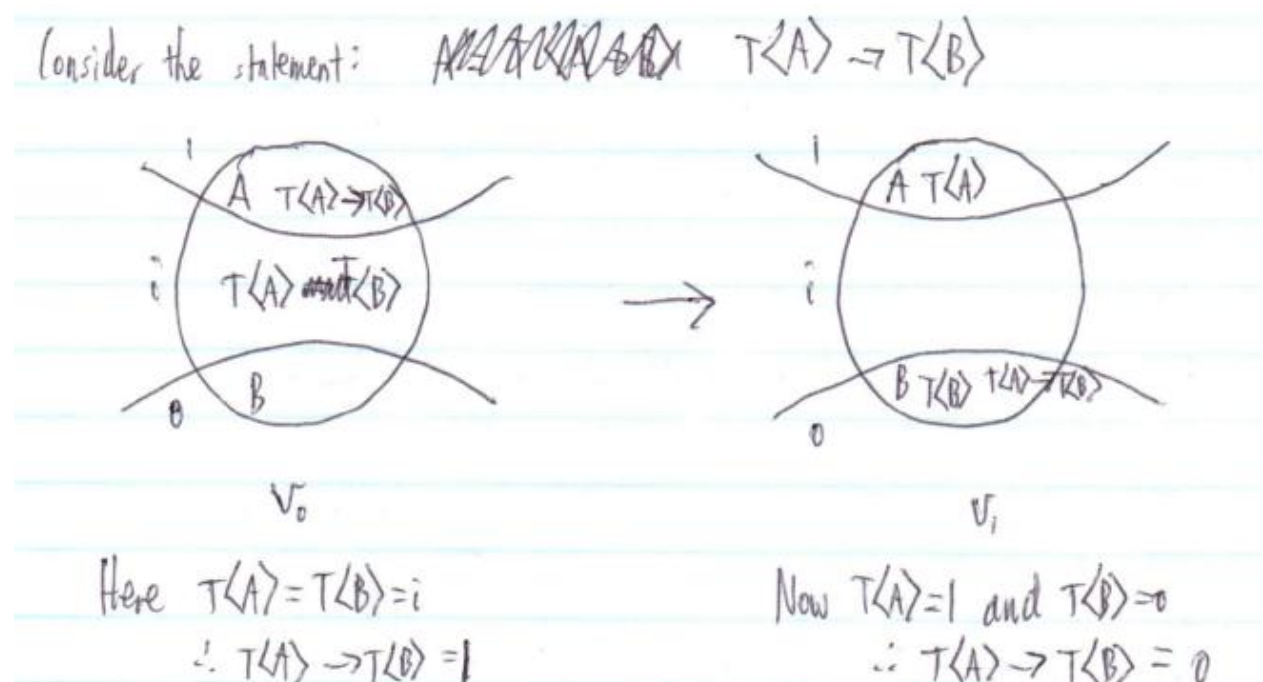
FACT: $v_i \subseteq v_{i+1}$ for each i .

The fixed point theorem states that eventually a fixed point is reached where no further propositions change in value, and at this point all statements of the form $T(A)$ will have the same value as A , meaning that Convention T is satisfied. At this fixed point, the liar statement will retain the initial value i .

There is a valuation v_k where $v_k = v_{k+1}$ and no new formulas are made true.

And $v_k(A) = v_k(T\langle A \rangle)$, since $v_k(A) = v_{k+1}(T\langle A \rangle) = v_k(T\langle A \rangle)$.

Representing this diagrammatically:



Curry's Paradox

This paradox highlights another problem with Convention T

“If this sentence is true, then p.”

It looks like this is a sentence π such that

$$T\pi \leftrightarrow (T\pi \rightarrow p)$$

But we can reason like this:

1. $T\pi \rightarrow (T\pi \rightarrow p)$ (Convention T)
2. $T\pi \rightarrow p$ (From 1, by *contraction*)
3. $(T\pi \rightarrow p) \rightarrow p$ (Convention T)
4. p (From 2, 3, by *modus ponens*)

But p is arbitrary. We can prove *anything*.

If we want Convention T we can't have *contraction* and *modus ponens* together.

Sorites Paradox

Let H_n be “ n grains of sand make a heap.” Then it seems like the following argument is paradoxical.

$$H_{1000}, H_{999} \supset H_{998}, H_{998} \supset H_{997}, \dots, H_2 \supset H_1, H_1 \supset H_0 \vdash H_0$$

There seem to be FOUR OPTIONS.

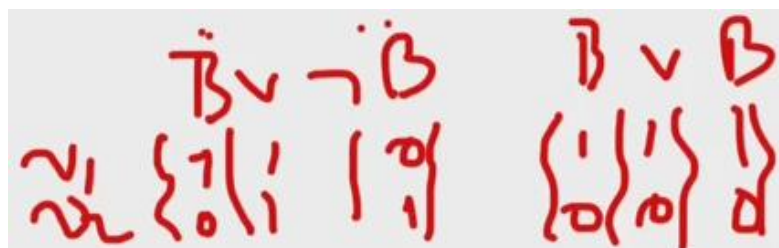
1. *Logic does not apply* to vague expressions.
2. Logic does apply, and the *argument is not valid*.
3. Logic does apply, the argument is valid but *one of the premises is false*.
4. Logic does apply, the argument is valid and sound, and therefore *the conclusion is true*.

One way to deal with this is by use of fuzzy logic (approach 2), which grants truth values between 0 and 1. Under this analysis, the sequence of inferences shown above is invalid.

Supervaluationism

In contrast to the epistemic conception of vagueness, a semantic conception such as supervaluationism treats the apparent semantic indeterminacy of vague predicates as real. Borderline cases, symptomatic of vagueness, are cases to which the predicate neither definitely applies nor definitely doesn't apply. Contra an epistemic account, the positive extension of a predicate is given by those objects to which the predicate definitely applies, the negative extension is given by those objects to which the predicate definitely does not apply, and the remaining (borderline) cases constitute the predicate's penumbra. This is called a fitting evaluation. Borderline cases are those with more than one fitting valuation.

Note that connectives under supervaluationism are not truth-functional, as their value depends on which valuations are being considered.



If, in general, something is true in all precisifications, supervaluationism describes it as "supertrue", while something false in all precisifications is described as "superfalse". One problematic aspect of supervaluationism is that it must commit to the fact that there is some cut-off point where true becomes false, even though there is no particular point of which it is true that it is the cut-off point. Since it is only this latter claim which is taken to commit one to the existence of a sharp boundary, it is argued, there is no commitment to there being such a boundary of which we are ignorant. With this explanation, however, doubts arise as to the adequacy of the logic, as we must also be prepared to admit that some existential statements can be true without having any true instance. In effect, the counter-intuitive aspects of the epistemic theory are avoided only at a cost to other intuitions.

Epistemicism

Epistemicism is a position about vagueness in the philosophy of language or metaphysics, according to which there are facts about the boundaries of a vague predicate which we cannot possibly discover. Given a vague predicate, such as 'is thin' or 'is bald', epistemicists hold that there is actually some sharp cut off, dividing cases where a person is actually thin from those in which they are not. Epistemicism gets its name because it holds that there is no semantic indeterminacy present in vague terms, only epistemic uncertainty. Epistemicism still uses the concept of multiple fitting evaluations, but rather than expressing different 'equally valid' ways of describing the world, these are taken to indicate situations which are equally consistent with our perceptual evidence. One of the valuations is really true, but we don't know (perhaps can't know) which it is.

LOGIC	SUPERVALUATIONS	EPISTEMICISM
evaluation	precisification	epistemic possibility
all fitting evaluations	the world	our possible evidence
true in all evaluations	super-true	known to be true
false in all evaluations	super-false	known to be false