Fourier Optics

Fourier Analysis

Introduction

Any periodic function can be written as the sum of harmonic basis functions - in the Fourier case we use sin and cos.

$$f(x+L) = f(x),$$
$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} [a_m \cos(mkx) + b_m \sin(mkx)]$$

We can use Fourier analysis to fine the coefficients a_m and b_m.

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} [a_m \cos(mkx) + b_m \sin(mkx)]$$
$$\int_0^L f(x) \cos(nkx)\, dx = \int_0^L \frac{a_0}{2} \cos(nkx)\, dx$$
$$+ \int_0^L \sum_{m=1}^{\infty} a_m \cos(mkx) \cos(nkx)\, dx + \int_0^L \sum_{m=1}^{\infty} b_m \sin(mkx) \cos(nkx)\, dx$$

By orthogonality and given that a single trig term integrated over its period is always zero:

$$\int_0^L f(x) \cos(nkx)\, dx = \sum_{m=1}^{\infty} \int_0^L a_m \cos(mkx) \cos(mkx)\, dx$$

By the inner product:

$$\int_0^L f(x) \cos(mkx)\, dx = \frac{a_m L}{2}$$
$$a_m = \frac{2}{L} \int_0^L f(x) \cos(mkx)\, dx$$

Similarly for the complex series:

$$f(x) = \sum_{m=-\infty}^{\infty} c_m e^{imkx}$$
$$\int_0^L f(x) e^{-imkx}\, dx = \int_0^L \sum_{m=-\infty}^{\infty} c_m e^{imkx} e^{-inkx}\, dx$$
$$\int_0^L f(x) e^{-imkx}\, dx = \int_0^L c_m e^{imkx} e^{-imkx}\, dx$$
$$\int_0^L f(x) e^{-imkx}\, dx = \frac{c_m L}{2}$$
$$c_m = \frac{2}{L} \int_0^L f(x) e^{-imkx}\, dx$$

Square Wave

Find the Fourier series coefficients for a square wave (note that $k = \frac{2\pi}{L} = 1$):

$$f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)\,dx \\ &= \frac{1}{\pi}\int_0^\pi 1\,dx \\ a_0 &= 1 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos(nkx)\,dx \\ &= \frac{1}{\pi}\int_0^\pi \cos(nkx)\,dx \\ &= \frac{1}{\pi}\left[\frac{1}{n}\sin nx\right]_0^\pi \\ a_n &= 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin(nkx)\,dx \\ &= \frac{1}{\pi}\int_0^\pi \sin(nx)\,dx \\ &= \frac{1}{\pi}\left[\frac{1}{n}\cos nx\right]_0^\pi \\ b_n &= \frac{1-(-1)^n}{n\pi} \end{aligned}$$

When dealing with aperiodic functions ($L \to \infty$), we need to instead use the inverse Fourier transform:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikx}\,dk$$

Gaussian

Find the Fourier transform of a Gaussian

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{x^2}{2\sigma^2}}$$

$$\begin{aligned} F(k) &= \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx \\ F(k) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{x^2}{2\sigma^2}}e^{-ikx}\,dx \\ &= \frac{1}{\sqrt{2\pi\sigma}}\int_{-\infty}^{\infty} \exp\left[-\left(\frac{x^2}{2\sigma^2} + ikx\right)\right]dx \\ &= \frac{1}{\sqrt{2\pi\sigma}}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma^2}(x^2 + 2\sigma^2 ikx)\right]dx \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma^2}(x^2 + (\sigma^2 ik)^2 - (\sigma^2 ik)^2 + 2\sigma^2 ikx)\right] dx$$
$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma^2}[x^2 + 2\sigma^2 ikx + (\sigma^2 ik)^2] + \frac{1}{2\sigma^2}(\sigma^2 ik)^2\right] dx$$
$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma^2}(x + \sigma^2 ik)^2 - \frac{1}{2}\sigma^2 k^2\right] dx$$
$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x+\sigma^2 ik)^2} e^{-\frac{1}{2}\sigma^2 k^2}\, dx$$
$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\sigma^2 k^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x+\sigma^2 ik)^2}\, dx$$

let $x_s = x + \sigma^2 ik$

$$= e^{-\frac{1}{2}\sigma^2 k^2} \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{x_s^2}{2\sigma^2}}\, dx_s$$
$$F(k) = e^{-\frac{\sigma^2 k^2}{2}}$$

Exponential

Find the Fourier transform of an exponential

$$f(x) = \frac{\gamma}{2} e^{-\gamma|x|}$$

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$
$$F(k) = \int_{-\infty}^{\infty} \frac{\gamma}{2} e^{-\gamma|x|} e^{-ikx}\, dx$$
$$= \int_{-\infty}^{0} \frac{\gamma}{2} e^{\gamma x} e^{-ikx}\, dx + \int_{0}^{\infty} \frac{\gamma}{2} e^{-\gamma x} e^{-ikx}\, dx$$
$$= \frac{\gamma}{2}\int_{-\infty}^{0} e^{(\gamma - ik)x}\, dx + \frac{\gamma}{2}\int_{0}^{\infty} e^{(-\gamma - ik)x}\, dx$$
$$= \frac{\gamma}{2}\left[\frac{1}{\gamma - ik} e^{(\gamma - ik)x}\right]_{-\infty}^{0} + \frac{\gamma}{2}\left[\frac{1}{-\gamma - ik} e^{(-\gamma - ik)x}\right]_{0}^{\infty}$$
$$= \frac{\gamma}{2}\left(\frac{1}{\gamma - ik}\right) + \frac{\gamma}{2}\left(\frac{1}{\gamma + ik}\right)$$
$$= \frac{\gamma}{2}\left[\frac{(\gamma - ik) + (\gamma + ik)}{(\gamma - ik)(\gamma + ik)}\right]$$
$$= \frac{\gamma}{2}\left[\frac{2\gamma}{\gamma^2 + k^2}\right]$$
$$F(k) = \frac{\gamma^2}{\gamma^2 + k^2}$$

Square

Find the Fourier transform of a square function

$$f(x) = \begin{cases} 1, & -a \le x \le a \\ 0, & \text{otherwise} \end{cases}$$

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

$$= \int_{-a}^{+a} e^{-ikx}\, dx$$

$$= \left[\frac{1}{-ik} e^{-ikx}\right]_{-a}^{+a}$$

$$= \frac{1}{-ik} e^{-ika} - \frac{1}{-ik} e^{ika}$$

$$= \frac{1}{ik} e^{ika} - \frac{1}{ik} e^{-ika}$$

$$= \frac{i}{k}\left(e^{ika} - e^{-ika}\right)$$

$$= \frac{2}{k}\sin(ka)$$

$$= \frac{2a}{ka}\sin(ka)$$

$$F(k) = 2a\,\mathrm{sinc}(ka)$$

Dirac Delta Function

The dirac delta function is a special type of function that satisfies the following properties:

$$\delta(x) = \begin{cases} 0, x \neq 0 \\ \infty, x = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

Physically it represents an idealized point mass or point charge. The delta function can also be defined as a limit, for example as the limiting distribution of the sequence of zero-centered normal distributions.

The dirac delta function has a number of important properties, including the projection property:

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0)$$

Sine Function

Find the Fourier transform of a sin function

$$f(x) = \sin(ax)$$

$$f(x) = -\frac{i}{2}\left(e^{iax} - e^{-iax}\right)$$

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

$$= -\int_{-\infty}^{\infty} \frac{i}{2}\left(e^{iax} - e^{-iax}\right) e^{-ikx}\, dx$$

$$F(k) = -\frac{i}{2}\int_{-\infty}^{\infty} e^{iax-ikx}\, dx + \frac{i}{2}\int_{-\infty}^{\infty} e^{-iax-ikx}\, dx$$

Use the integral form of the delta-function:

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-a)}\, dk$$

Or swapping the variables:

$$\delta(k-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix(k-a)}\,dx$$

Substituting this expression in:

$$F(k) = -\frac{i}{2}\int_{-\infty}^{\infty} e^{ix(a-k)}\,dx + \frac{i}{2}\int_{-\infty}^{\infty} e^{-ix(a+k)}\,dx$$
$$= -\frac{i}{2}2\pi\delta(a-k) + \frac{i}{2}2\pi\delta(-a-k)$$
$$= -i\pi\delta(a-k) + i\pi\delta(-a-k)$$

Since $\delta(x) = \delta(-x)$:

$$F(k) = i\pi\delta(a+k) - i\pi\delta(k-a)$$

Wave Equation

Any function of the form $f(x - vt + \phi_0)$ is a travelling wave with speed v.

Such a waveform must satisfy the wave equation:

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

One simply functional form that satisfies these properties is:

$$\psi(x,t) = A\sin(kx - \omega t)$$

Where $k = \frac{2\pi}{\lambda}$, $\omega = 2\pi f$

This can also be written in complex form:

$$\psi(x,t) = Ae^{kx-\omega t} = Ae^{i\phi}$$

When dealing with complex wave equations in this manner, we simply do all the calculations with the entire complex number, and then ignore the imaginary part.

We can extend this analysis into three dimensions by use of the plane wave:

$$\psi(x,y,z,t) = A\sin\left(k_x x + k_y y + k_z z - \omega t\right)$$
$$\psi(x,y,z,t) = A\sin\left(\tilde{k}\cdot\tilde{r} - \omega t\right)$$

The surfaces of a plane wave simply represent the joining of all points of equal phase. In defining a plane wave, we always implicitly define its direction of travel, which is the direction of the propagation vector $\tilde{k}$.

The three-dimensional wave equation is written in the form:

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

The fundamental underlying principle of Fourier optics is that all complex optical phenomena of interest can be represented by the superposition of plane waves.

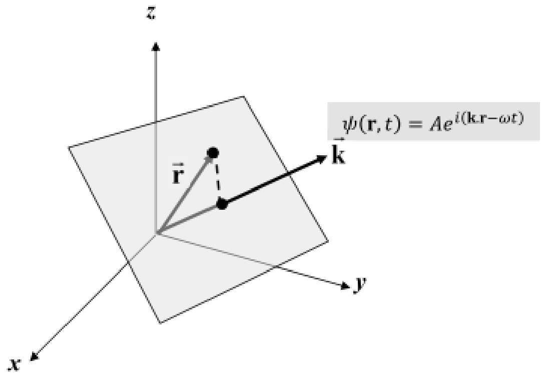
Wave Optics

Plane Waves

One particularly useful and important way of representing three dimensional waves is called a plane wave. A plane wave is represented by the equation:

$$\psi(r,t) = Ae^{i(k_x x + k_y y + k_z z)} = Ae^{i(k\cdot r)}$$

The vector $\tilde{k}$ represents the vector normal to the surface of the plane, and hence shows the direction the wave is travelling.



As an example, a plane wave travelling in the direction $(1,2,3)$ would yield:

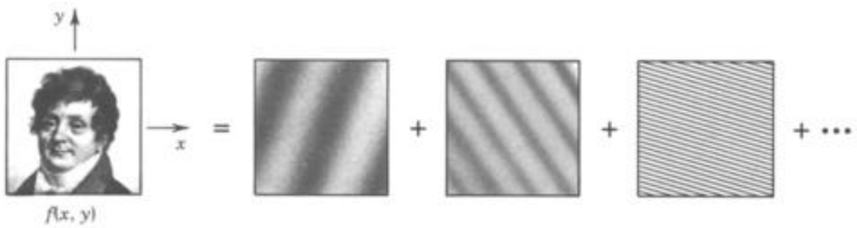
$$\tilde{k} = k(1,2,3)$$
$$|\tilde{k}| = \sqrt{1^2 + 2^2 + 3^2}$$
$$= \sqrt{1+4+9}$$
$$|\tilde{k}| = \sqrt{14}$$
$$\tilde{k} = \frac{k}{\sqrt{14}}(1,2,3)$$
$$\tilde{r} = x\hat{x} + y\hat{y} + z\hat{z}$$
$$\tilde{k}\cdot\tilde{r} = \frac{k}{\sqrt{14}}(1,2,3)\cdot(x\hat{x} + y\hat{y} + z\hat{z})$$

Hence we have:

$$\psi(r,t) = A\sin\left(\frac{k}{\sqrt{14}}(x + 2y + 3z) \pm vt\right)$$

Plane waves are especially useful because by taking the Fourier transform of any wavefunction, we are able to represent that wave in terms of plane waves, with the amount of each plane wave given by the Fourier coefficients $F(k)$.

This leads to the fundamental insight of Fourier optics: that any image, however complex, can be represented as an infinite sum of plane waves. This is very counterintuitive, but Fourier Series theory tells us it is so!

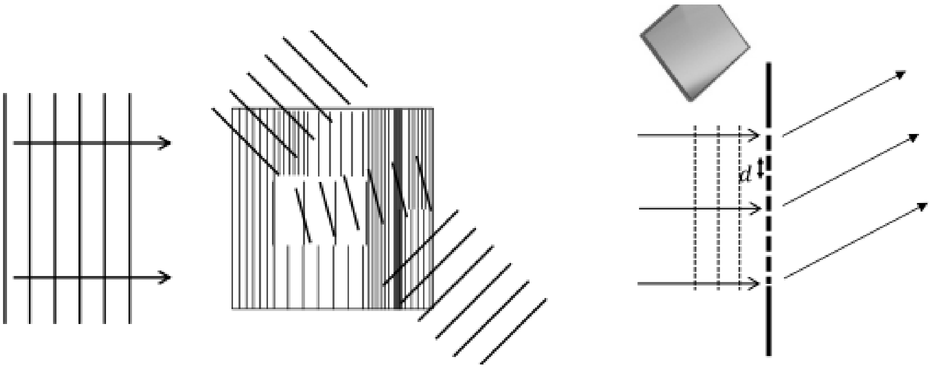


Spatial Frequency

Spatial frequency is defined as the number of wavelengths in a unit of distance. If we consider diffraction gratings, each with a different spatial frequency (different distance between the slits), each will diffract incident light at a different angle. This is a result of the diffraction equation:

$$\sin\theta = \frac{\lambda}{d}$$

This means that if we have a complex image that is comprised of a complex superposition of diffraction gratings (as per Fourier series theory), then incident light on this diffraction pattern will be split up into plane waves travelling in different directions. Each different wave direction (and hence a different k vector) corresponds to a unique spatial frequency in the diffraction pattern that produced that refracted plane wave.



This leads to the second core principle of Fourier optics:
A transverse spatial variation (at z=0) with spatial frequencies v_x, v_y corresponds to a plane wave in a specific direction.

Propagating a Plane EM Wave

1. Begin with initial electric field function $E(X,Y,z)$ at point $z = 0$
2. Find the Fourier transform of the electric field:

$$\tilde{E}\left(k_x, k_y, z=0\right) = \iint_{-\infty}^{+\infty} E(x,y,z=0)e^{-i(k_x x + k_y y)}\,dxdy$$

3. Multiply this Fourier transform by the free-space transfer function $e^{ik_z z}$ to find the Fourier transform of the electric field at z:

$$\tilde{E}\left(k_x, k_y, z\right) = e^{ik_z z}\tilde{E}\left(k_x, k_y, z=0\right)$$

4. Find the inverse Fourier transform of the propagated electric field $\tilde{E}'$.

$$E(X,Y,z) = F^{-1}\left[F[E(x,y,z=0)]e^{ik_z z}\right]$$

$$E(X,Y,z) = \frac{1}{4\pi^2}\iint_{\infty}^{\infty} \tilde{E}\left(k_x, k_y, z=0\right)e^{ik_z z}e^{i(k_y x + k_y y)}\,dk_x dk_y$$

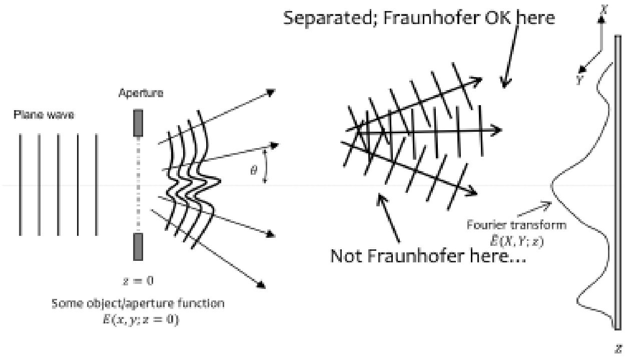
Summary of Fourier Optics

1. Plane waves propagate in straight lines
2. Gratings of spatial frequency diffract in a particular direction
3. Can describe object (lightfield) as superposition of “gratings” (spatial frequency components)
4. Find superposition from Fourier transform of object or grating function
5. Fourier transform = “how much” of each grating is present
6. Objects diffract into different directions according to “how much” of each spatial frequency
7. Propagated wavefield is then sum of all those diffracted plane waves

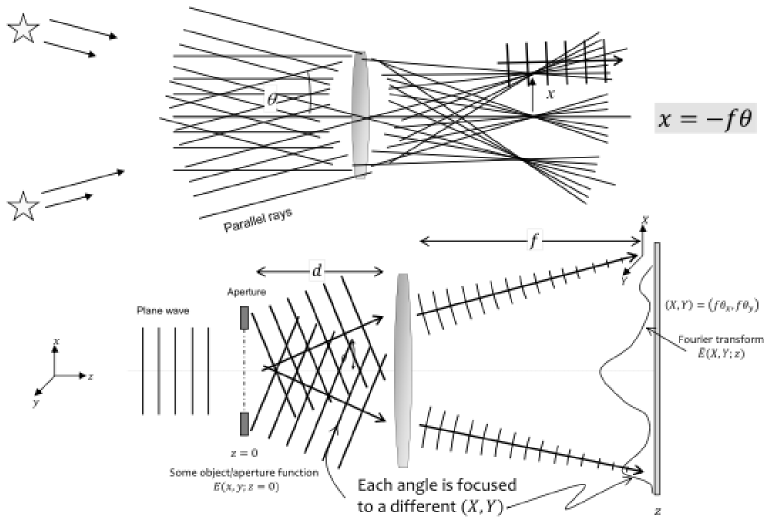
Fraunhofer Diffraction

Introduction

The Fraunhofer diffraction equation is used to model the diffraction of waves when the diffraction pattern is viewed at a long distance from the diffracting object. An intuitive way of thinking about this is that Fraunhofer diffraction describes the field pattern when the diffracted light has been able to travel far enough such that the different plane waves have been able to separate.



Rather than having to place our screen an infinite distance away from the diffraction grating, we can instead observe the Fraunhofer diffraction pattern by placing a screen at the far focal point of a converging lens. A lens acts as a Fourier transform computer.



The Fraunhofer diffraction is calculated using the formula (note: if using a lens then $z = f$):

$$E(X,Y,z) = \frac{1}{\lambda z}\tilde{E}\left(\frac{k_x X}{z}, \frac{k_y Y}{z}\right)$$

Single Slit

Find the Fraunhofer diffraction pattern for a single narrow slit

$$E(x) = A_0 \; for \; |x| < \frac{b}{2}$$

$$E(X,z) = \frac{1}{\lambda z}\tilde{E}\left(\frac{k_x X}{z}\right)$$

$$E(X,z) = \frac{A_0 b}{\lambda z}\operatorname{sinc}\left(\frac{\pi X b}{\lambda z}\right)$$

Double Slit

Represent each slit as a delta-function.

$$E(x, z=0) = A_0 \left(\delta \left(x - \frac{a}{2} \right) + \delta \left(x + \frac{a}{2} \right) \right)$$

$$E(X, z) = \frac{1}{\lambda z} \tilde{E} \left(\frac{k_x X}{z} \right)$$

$$E(X, z) = \frac{1}{\lambda z} 2A_0 \cos \left(\frac{\pi X a}{\lambda z} \right)$$

Sinusoidal Grating

Find the Fraunhofer diffraction pattern for a sinusoidal diffraction grating

$$E(x, z=0) = E_0 \cos^2 gx$$

$$E(x, z=0) = \frac{E_0}{2} + \frac{E_0}{4} (e^{2igx} + e^{-2igx})$$

$$E(X, z) = \frac{1}{\lambda z} \tilde{E} \left(\frac{k_x X}{z} \right)$$

$$E(X, z) = \frac{E_0}{2} \lambda \delta(\theta_x) + \frac{E_0}{4} \lambda \left(\delta \left(\theta_x - \frac{g\lambda}{\pi} \right) + \delta \left(\theta_x + \frac{g\lambda}{\pi} \right) \right)$$

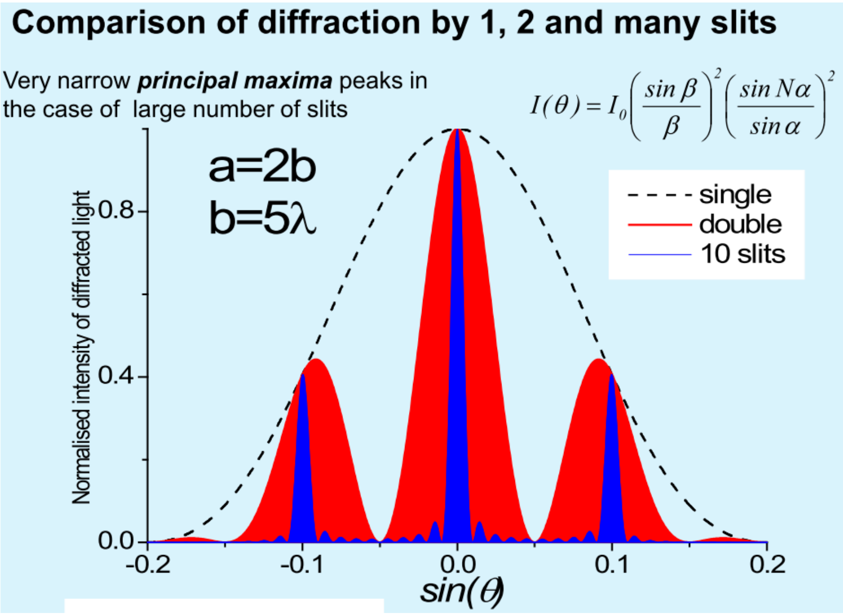
Array Theorem

The array theorem states that the field distribution in the Fraunhofer diffraction pattern of an array of similarly oriented identical apertures is given by the product of the Fourier transform of a single one of these aperture functions T_1, and the pattern that would result from a set of point sources (delta functions) arranged in the same configuration. For a set of points in an x-y plane this pattern of delta functions is given by:

$$T_\delta = \sum_{j=1}^{\infty} \delta(x - x_j)\delta(y - y_i)$$

Hence the total field distribution is given by:

$$E(x, y) = F(T_1)F(T_\delta)$$



Further Optics Principles

Optical Resolution

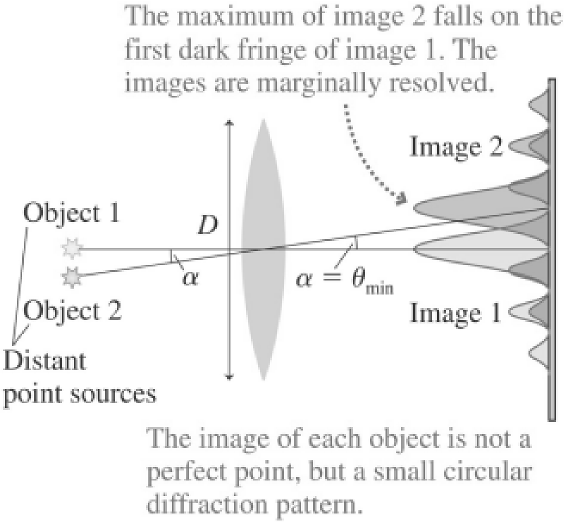
Two objects or 'spots' near each other will both produce diffraction rings. If the objects are sufficiently close together, these diffraction rings will overlap, producing a blurred image. Rayleigh's criterion states that we can 'just resolve' two spots if the centre of one refracted Airy disc falls on the first minimum of the second refracted Airy disc.

The angular width of a central lobe is given by:

$$\Delta\theta = 2.44\frac{\lambda}{D}$$

Hence the Rayleigh criterion tells us that two objects will be just resolvable at the angle:

$$\theta_{min} = 1.22\frac{\lambda}{D}$$



Fresnel Diffraction

While Frauhofer diffraction applies to the far field (away from the object/lens), Fresnel diffraction applies to the near field (close to the object/lens).

$$E(X,Y,z) = F^{-1}\left[\tilde{E}(k_x,k_y,z=0)e^{ik_z z}\right]$$

Using the paraxial approximation for the propagation function:

$$e^{ikz} = e^{iz\left(k^2-(k_x^2+k_y^2)\right)^{\frac{1}{2}}} = e^{izk\left(1-\frac{1}{k^2}(k_x^2+k_y^2)\right)^{\frac{1}{2}}} \approx e^{i\left(zk-\frac{z}{2k}(k_x^2+k_y^2)\right)} = e^{ikz}e^{-\frac{iz}{2k}(k_x^2+k_y^2)}$$

Hence the Fresnel diffraction becomes:

$$E(X,Y,z) = F^{-1}\left[F[E(x,y,z=0)]e^{ikz}e^{-\frac{iz}{2k}(k_x^2+k_y^2)}\right]$$

Notice that the second term in the square brackets is actually the Fourier transform of a Gaussian

$$F^{-1}\left[e^{-\frac{iz}{2k}(k_x^2+k_y^2)}\right] = \frac{1}{i\lambda z}e^{\frac{ik}{2z}(x^2+y^2)}$$

$$E(X,Y,z) = F^{-1}\left[F[E(x,y,z=0)]F\left[e^{ikz}\frac{1}{i\lambda z}e^{\frac{ik}{2z}(x^2+y^2)}\right]\right]$$

By the Convolution theorem, we know that the inverse Fourier transform of the product of two Fourier transforms can be written as the convolution of the two original functions:

$$E(X,Y,z) = E(x,y,z=0) \otimes e^{ikz} \frac{1}{i\lambda z} e^{\frac{ik}{2z}(x^2+y^2)}$$

$$E(X,Y,z) = \iint E(x,y) e^{ikz} \frac{1}{i\lambda z} e^{\frac{ik}{2z}((X-x)^2+(Y-y)^2)}\, dxdy$$

In the far field, $x^2 \ll \lambda z$, $X^2 \ll \lambda z$, hence:

$$e^{\frac{ik}{2z}(x^2+y^2)} \approx e^0 = 1$$

$$e^{\frac{ik}{2z}(X^2+Y^2)} \approx e^0 = 1$$

$$e^{\frac{ik}{2z}((X-x)^2+(Y-y)^2)} \approx e^{\frac{ik}{2z}(xX+yY)}$$

$$E(X,Y,z) = \iint E(x,y) \frac{e^{ikz}}{i\lambda z} e^{\frac{ik}{z}(xX+yY)}\, dxdy$$

Substitute $k_x = \frac{kX}{z}, k_y = \frac{kY}{z}$:

$$E(X,Y,z) = \frac{e^{ikz}}{i\lambda z} \iint E(x,y) e^{ik(k_x x + k_y y)}\, dxdy$$

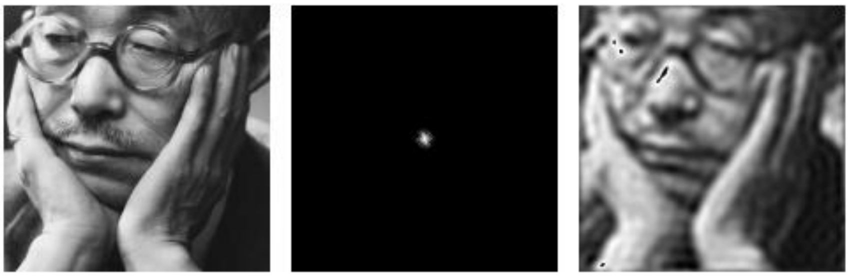
$$E(X,Y,z) = \frac{e^{ikz}}{i\lambda z} \tilde{E}\left(\frac{kX}{z}, \frac{kY}{z}\right)$$

Spatial Filtering

Recall that a lens acts as a Fourier transform computer, separating out the different spatial frequencies of an image. Specifically, high spatial frequency waves (corresponding to fine details) are diffracted to a larger angle, meaning that these waves are represented around the edges of the diffraction pattern. Conversely, low spatial frequency waves (corresponding to basic shapes and textures) are diffracted less, and so they are represented near the centre of the diffraction pattern.

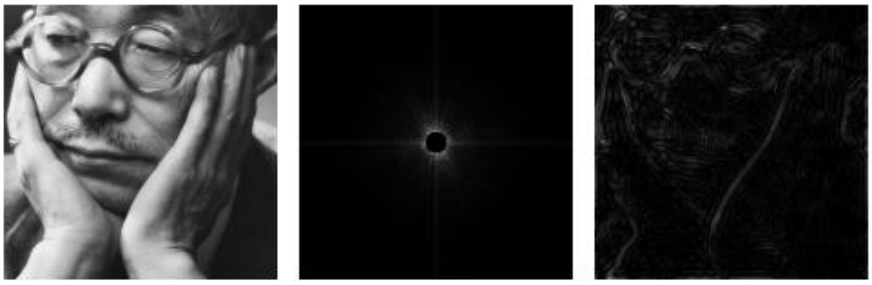
This separation means that by selective removal of particular parts of the diffraction pattern, we are able to remove particular spatial frequency components of the image. If we then take the inverse Fourier transform of this altered diffraction pattern, we will have the original image with the relevant spatial frequencies absent.

Low-pass filter: removes high frequencies near the edge, and transmits only low frequencies



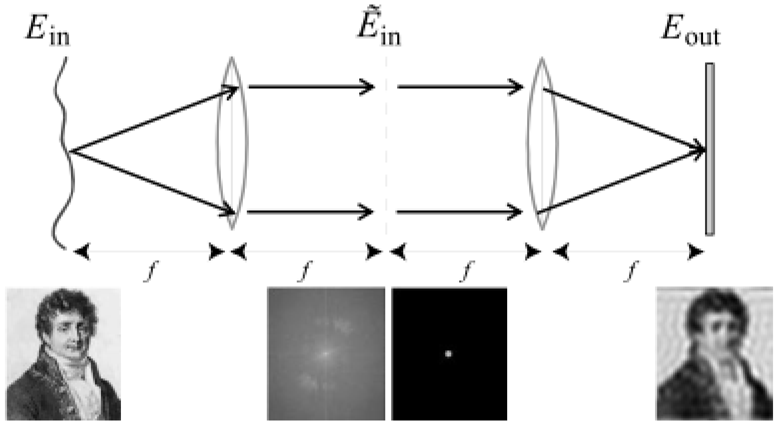
(a) Input image (b) Fourier transform amplitude, has been low pass filtered (c) Reconstruction of image

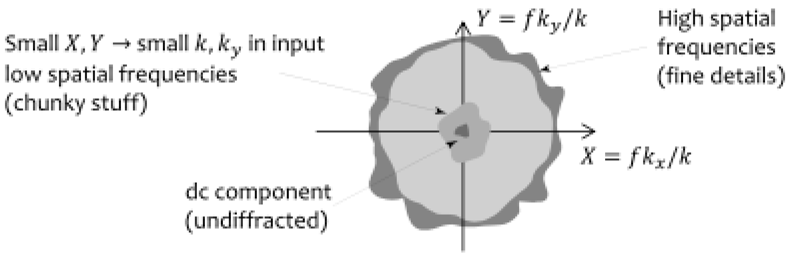
High-pass fi Figure 4.6: The effect of low-pass filtering on the image quencies



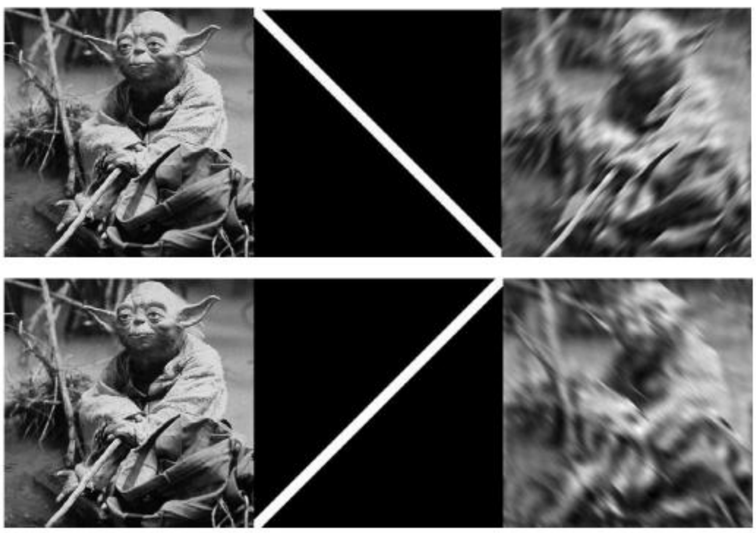
(a) Input image (b) Fourier transform amplitude, has been high-pass filtered (c) Reconstruction of image

Figure 4.7: The effect of high-pass filtering on the image





Directional filtering: block all lines in one particular direction



Convolution

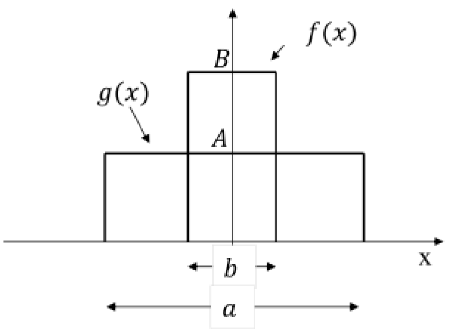
A convolution is an integral that expresses the amount of overlap of one function g as it is shifted over another function f. It therefore "blends" one function with another. The formula for the convolution integral is given by:

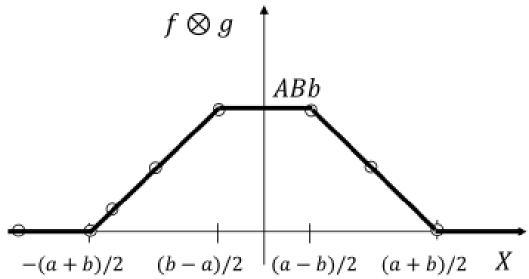
$$h(x) = \int_{-\infty}^{+\infty} f(x)g(X - x)\, dx$$

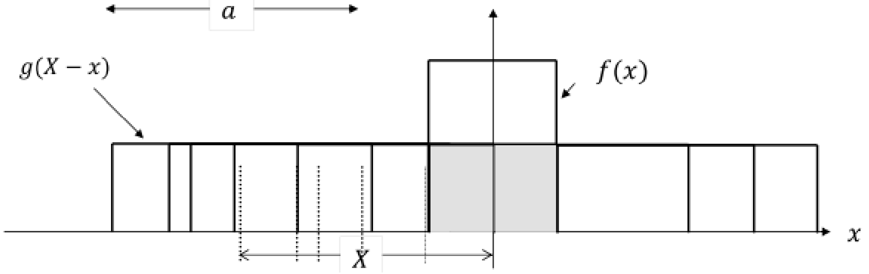
This can also be written as:

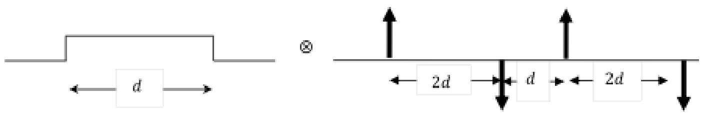
$$h = f \otimes g$$

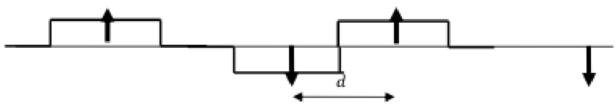
To sketch a convolution, take the 'moving' function $g(X - x)$, and move it across the 'stationary' function $f(x)$. At each value of x, determining the area of overlap for the two functions. This area will become the value of the convolution $h(x)$ at that particular x value.











There is a very useful theorem called the Convolution Theorem, which states that the Fourier transform of the convolution of two functions is equation to the product of the Fourier transform of those functions. This is written as:

$$F\left[\int_{-\infty}^{+\infty} f(x)g(X - x)\, dx\right] = F(k)G(k)$$

Cross-correlation is a measure of how similar two functions are. It is defined almost in exactly the same way as convolution - the only difference being that the complex conjugate of $f(x)$:

$$h(x) = \int_{-\infty}^{+\infty} f^*(x)g(X - x)\, dx$$

Sampling Theorem

If a continuous signal is sampled at a rate greater than twice its highest frequency component, then it is possible to recover the signal from its samples.