Differential Equations

Ordinary Differential Equations

First Order Homogenous Linear ODE

To solve equations of the form:

$$w'(t) - aw(t) = 0$$

Simply re-arrange and solve by direct integration:

$$w'(t) = aw(t)$$
$$w(t) = a\int w(t)\,dt$$

First Order Inhomogenous Linear ODE

To solve equations of the form:

$$w'(t) + p(t)w(t) = r(t)$$

Use the method of reduction to quadratures. Consider the following example:

$$w'(t) - \frac{2}{t}w(t) = 1$$

Define integrating factor

$$\text{let } I(t) = e^{\int p(t)dt}$$
$$I(t) = e^{-2\int \frac{1}{t}dt}$$
$$I(t) = e^{-2\log(t)}$$
$$I(t) = e^{\log(t^{-2})}$$
$$I(t) = \frac{1}{t^2}$$

Hence

$$\frac{d\big(I(t)w(t)\big)}{dt} = I(t)$$
$$\frac{d\left(\frac{1}{t^2}w(t)\right)}{dt} = \frac{1}{t^2}$$
$$\int d\left(\frac{1}{t^2}w(t)\right) = \int \frac{1}{t^2}dt$$
$$\frac{1}{t^2}w(t) = -\frac{1}{t} + C$$
$$w(t) = -t + Ct^2$$

Second Order Homogenous Constant Coefficient ODE

To solve equations of the form:

$$au''(x) + bu'(x) + cu(x) = 0$$

Simply use the trial substitution:

$$w(x) = e^{\lambda x}$$

And solve the resulting characteristic equation:

$$a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0$$
$$a\lambda^2 + b\lambda + c = 0$$

Second Order Inhomogenous Linear ODE

There is no simple way to solve an equation of the form:

$$w''(t) + p(t)w'(t) + q(t)w(t) = r(t)$$

We know that the ODE will have solutions so long as $p(t), q(T)$ and $r(t)$ are continuous. If we can find one solution to the homogenous version of the equation, we can use reduction of order to find a full solution.

Consider Bessel's Equation of order $\frac{1}{2}$:

$$t^2 w''(t) + tw'(t) + \left(t^2 - \left(\frac{1}{2}\right)^2\right) w(t) = 0$$

Given $w_0(t) = t^{-\frac{1}{2}} \sin(t)$ is a solution, find the general solution

Make the substitution:

$$w(t) = u(t)w_0(t)$$
$$w(t) = u(t)t^{-\frac{1}{2}} \sin(t)$$
$$w'(t) = -\frac{1}{2}t^{-\frac{3}{2}} \sin(t)\, u(t) + t^{-\frac{1}{2}} \cos(t)\, u(t) + t^{-\frac{1}{2}} \sin(t)\, u'(t)$$
$$\begin{aligned} w''(t) = \frac{3}{4}t^{-\frac{3}{2}} \sin(t)\, u(t) - \frac{1}{2}t^{-\frac{3}{2}} \cos(t)\, u(t) - \frac{1}{2}t^{-\frac{3}{2}} \sin(t)\, u'(t) - \frac{1}{2}t^{-\frac{3}{2}} \cos(t)\, u(t) \\ - t^{-\frac{1}{2}} \sin(t)\, u(t) + t^{-\frac{1}{2}} \cos(t)\, u'(t) - \frac{1}{2}t^{-\frac{3}{2}} \sin(t)\, u'(t) + t^{-\frac{1}{2}} \cos(t)\, u'(t) \\ + t^{-\frac{1}{2}} \sin(t)\, u''(t) \end{aligned}$$
$$\begin{aligned} w''(t) = \frac{3}{4}t^{-\frac{3}{2}} \sin(t)\, u(t) - t^{-\frac{3}{2}} \cos(t)\, u(t) - t^{-\frac{1}{2}} \sin(t)\, u(t) + 2t^{-\frac{1}{2}} \cos(t)\, u'(t) \\ - t^{-\frac{3}{2}} \sin(t)\, u'(t) + t^{-\frac{1}{2}} \sin(t)\, u''(t) \end{aligned}$$

Hence we have:

$$\begin{aligned} t^2 \left[\frac{3}{4}t^{-\frac{3}{2}} \sin(t)\, u(t) - t^{-\frac{3}{2}} \cos(t)\, u(t) - t^{-\frac{1}{2}} \sin(t)\, u(t) + 2t^{-\frac{1}{2}} \cos(t)\, u'(t) - t^{-\frac{3}{2}} \sin(t)\, u'(t) \right. \\ \left. + t^{-\frac{1}{2}} \sin(t)\, u''(t)\right] + t\left[-\frac{1}{2}t^{-\frac{3}{2}} \sin(t)\, u(t) + t^{-\frac{1}{2}} \cos(t)\, u(t) + t^{-\frac{1}{2}} \sin(t)\, u'(t)\right] \\ + \left(t^2 - \left(\frac{1}{2}\right)^2\right)\left[u(t)t^{-\frac{1}{2}} \sin(t)\right] = 0 \end{aligned}$$
$$\begin{aligned} \frac{3}{4}t^{\frac{1}{2}} \sin(t)\, u(t) - t^{\frac{1}{2}} \cos(t)\, u(t) - t^{\frac{3}{2}} \sin(t)\, u(t) + 2t^{\frac{3}{2}} \cos(t)\, u'(t) - t^{\frac{1}{2}} \sin(t)\, u'(t) \\ + t^{\frac{3}{2}} \sin(t)\, u''(t) - \frac{1}{2}t^{-\frac{1}{2}} \sin(t)\, u(t) + t^{\frac{1}{2}} \cos(t)\, u(t) + t^{\frac{1}{2}} \sin(t)\, u'(t) \\ + u(t)t^{\frac{3}{2}} \sin(t) - \frac{1}{4}u(t)t^{-\frac{1}{2}} \sin(t) = 0 \end{aligned}$$

Note that the terms with $u(t)$ should cancel out:

$$\left(\frac{3}{4}t^{\frac{1}{2}}\sin(t)\,u(t)-\frac{1}{2}t^{-\frac{1}{2}}\sin(t)\,u(t)-\frac{1}{4}t^{-\frac{1}{2}}\sin(t)\,u(t)\right)+\left(-t^{\frac{1}{2}}\cos(t)\,u(t)+t^{\frac{1}{2}}\cos(t)\,u(t)\right)$$
$$+\left(-t^{\frac{3}{2}}\sin(t)\,u(t)+u(t)t^{\frac{3}{2}}\sin(t)\right)+2t^{\frac{3}{2}}\cos(t)\,u'(t)$$
$$+\left(-t^{\frac{1}{2}}\sin(t)\,u'(t)+t^{\frac{1}{2}}\sin(t)\,u'(t)\right)+t^{\frac{3}{2}}\sin(t)\,u''(t)=0$$
$$2t^{\frac{3}{2}}\cos(t)\,u'(t)+t^{\frac{3}{2}}\sin(t)\,u''(t)=0$$
$$2\cos(t)\,u'(t)+\sin(t)\,u''(t)=0$$
$$u''(t)+2\cot(t)\,u'(t)=0$$

Note that this is actually a first order ODE in disguise:

$$u''(t)+2\cot(t)\,u'(t)=0$$
$$m'(t)+2\cot(t)\,m(t)=0$$
$$I(t)=e^{\int 2\cot(t)dt}$$
$$I(t)=e^{2\log[\sin(t)]}$$
$$I(t)=\sin^2(t)$$

Hence:

$$\frac{d[\sin^2(t)\,m(t)]}{dt}=0$$
$$\sin^2(t)\,m(t)=C$$
$$\sin^2(t)\,u'(t)=C$$
$$u'(t)=\frac{C}{\sin^2(t)}$$
$$u(t)=C\int \operatorname{cosec}^2(t)\,dt$$
$$u(t)=C[-\cot(t)+D]$$
$$u(t)=C_1\cot(t)+C_2$$

We can know find the full solution of the original ODE:

$$w(t)=u(t)w_0(t)$$
$$w(t)=(C_1\cot(t)+C_2)\left(t^{-\frac{1}{2}}\sin(t)\right)$$
$$w(t)=C_1t^{-\frac{1}{2}}\cos(t)+C_2t^{-\frac{1}{2}}\sin(t)$$

We now need to check whether our two solutions are linearly independent (if not we don't need both of them). To do this we construct what is called the Wronskian, which is the determinant:

$$W(t)=\begin{vmatrix} w_1(t) & w_2(t) \\ w_1'(t) & w_2'(t) \end{vmatrix}$$
$$W(t)=w_1(t)w_2'(t)-w_2(t)w_1'(t)$$

In this particular case:

$$W(t)=\sin(t)\,t^{-\frac{1}{2}}\left[-\frac{1}{2}\cos(t)\,t^{-\frac{3}{2}}-\sin(t)\,t^{-\frac{1}{2}}\right]-\cos(t)\,t^{-\frac{1}{2}}\left[-\frac{1}{2}\sin(t)\,t^{-\frac{3}{2}}+\cos(t)\,t^{-\frac{1}{2}}\right]$$
$$W(t)=-\frac{1}{2}\cos(t)\sin(t)\,t^{-2}-\frac{\sin^2(t)}{t}+\frac{1}{2}\cos(t)\sin(t)\,t^{-2}-\frac{\cos^2(t)}{t}$$
$$W(t)=-\frac{\sin^2(t)}{t}-\frac{\cos^2(t)}{t}$$

$$W(t) = -\frac{1}{t}[\sin^2(t) + \cos^2(t)]$$
$$W(t) = -\frac{1}{t}$$

This is always non-zero for all t, hence we know that the two solutions are linearly independent.

What if we do not have an initial solution available? Then obviously we cannot use reduction of order - we'll have to try something else. Often the best one can do is check to see if the equation at hand is one of a few that are encountered particularly often, and have known solutions. A number of these are examined below.

Airy's Equation

Airy's equation has the form:

$$u''(x) = xu(x)$$
$$u''(x) - xu(x) = 0$$

It is perhaps somewhat surprising that even this relatively simple equation has no simple solutions. The known solutions are denoted Airy functions of the first kind $Ai(x)$ and of the second kind $Bi(x)$.

Cauchy-Euler Equation

The Cauchy-Euler equation has the form:

$$\alpha t^2 w''(t) + \beta t w'(t) + \gamma w(t) = 0$$

Where α, β, γ are constants. To solve this equation we make the trial solution $w(t) = t^\mu$:

$$[\alpha\mu(\mu - 1) + \beta\mu + \gamma]t^\mu = 0$$
$$\alpha\mu^2 + (\beta - \alpha)\mu + \gamma = 0$$

If this equation has two distinct roots μ_1 and μ_2 then:

$$w(t) = At^{\mu_1} + Bt^{\mu_2}$$

Otherwise the solution is:

$$w(t) = A\log(t)\, t^{\mu_1} + Bt^{\mu_1}$$

If the only solution is the trivial solution then we simply have:

$$w(t) = A\log(t) + B$$

Legendre's Equation

Legendre's equation has the form:

$$(1 - x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + v(v + 1)u = 0$$

Where $v \geq -\frac{1}{2}$. If v is restricted to be integers, then the only well-behaved solutions are multiples of the Legendre polynomials $P_n(x)$. Legendre polynomials form a vector space.

Bessel's Equation

Bessel's equation of order v is:

$$r^2 w''(r) + rw'(r) + (r^2 - v^2)w(r) = 0$$

There are two classes of solutions to Bessel's equation: those that are finite at the origin (denoted $J_v(r)$), and those with a singularity at the origin (denoted $Y_v(r)$).

Linear Systems

Both differential equations with constant coefficients of order greater than two, and also linear systems of first or second (or higher) order differential equations, can be solved using a technique involving eigenvalues and fundamental matrices. The method is best illustrated with an example.

Consider the following system of differential equations:

$$\begin{aligned} x'(t) &= 3x(t) + e^{3t} \\ y'(t) &= 2x(t) - y(t) - 2z(t) \\ z'(t) &= 3x(t) + 6y(t) + 6z(t) \end{aligned}$$

The first step is to write the system in matrix form:

$$\begin{aligned} x'(t) &= 3x(t) + 0y(t) + 0z(t) + e^{3t} \\ y'(t) &= 2x(t) - 1y(t) - 2z(t) \\ z'(t) &= 3x(t) + 6y(t) + 6z(t) \end{aligned}$$

$$\tilde{x}'(t) = A(t)\tilde{x} + f(t)$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & -2 \\ 3 & 6 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} e^{3t} \\ 0 \\ 0 \end{bmatrix}$$

If we can solve the homogenous version of this equation, we could find the general solution by reduction to quadratures. So for now consider the homogenous problem:

$$\tilde{x}'(t) - A(t)\tilde{x} = \tilde{0}$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & -2 \\ 3 & 6 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Try the solution (where $\tilde{v}$ is a vector of as-yet-unknown constants):

$$\tilde{x}(t) = e^{\lambda t}\tilde{v}$$

Hence we have:

$$\begin{aligned} \lambda e^{\lambda t}\tilde{v} &= A(t)e^{\lambda t}\tilde{v} \\ A(t)\tilde{v} &= \lambda\tilde{v} \end{aligned}$$

This last equation is of the form of a matrix times a vector, yielding the original vector multiplied by a constant. This should sound familiar - this is an eigenvalue problem! Specifically, $\tilde{v}$ is an eigenvector of $A(t)$ with the eigenvalue λ. Now that we know the solutions to the homogenous equation are directly related to the eigenvalues and corresponding eigenvectors of A, we just need to find them:

$$\begin{aligned} A(t)\tilde{v} &= \lambda\tilde{v} \\ (A(t) - \lambda I)\tilde{v} &= 0 \\ A(t) - \lambda I &= \begin{bmatrix} 3-\lambda & 0 & 0 \\ 2 & -1-\lambda & -2 \\ 3 & 6 & 6-\lambda \end{bmatrix} \\ \det(A(t) - \lambda I) &= (3-\lambda)[(-1-\lambda)(6-\lambda) - (6)(-2)] - 0 + 0 \end{aligned}$$

Since we want linearly independent solutions the determinant must be zero:

$$(3-\lambda)(\lambda^2-5\lambda+6)=0$$
$$(3-\lambda)(\lambda-3)(\lambda-2)=0$$
$$-(\lambda-3)^2(\lambda-2)=0$$
$$\lambda=3,\lambda=2$$

In this case we have found only two distinct eigenvalues. This is unfortunate, as it means we will have some extra work to do. First, however, we need to find the eigenvectors corresponding to these eigenvalues:

$$A(t)\tilde{v}=\lambda\tilde{v}$$
$$(A(t)-\lambda I)\tilde{v}=\tilde{0}$$
$$\left(\begin{bmatrix}3 & 0 & 0\\ 2 & -1 & -2\\ 3 & 6 & 6\end{bmatrix}-\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}\right)\tilde{v}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$\begin{bmatrix}1 & 0 & 0\\ 2 & -3 & -2\\ 3 & 6 & 4\end{bmatrix}\tilde{v}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Writing this in augmented matrix form:

$$\begin{bmatrix}1 & 0 & 0 & | & 0\\ 2 & -3 & -2 & | & 0\\ 3 & 6 & 4 & | & 0\end{bmatrix}$$

Solving using row reduction (steps omitted):

$$\begin{bmatrix}1 & 0 & 0 & | & 0\\ 0 & 3 & 2 & | & 0\\ 0 & 0 & 0 & | & 0\end{bmatrix}$$

Thus:

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Hence $v_1=0$, $v_3=t$, and v_2:

$$3v_2+2v_3=0$$
$$3v_2=-2t$$
$$v_2=-\frac{2}{3}t$$
$$\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\-\frac{2}{3}\\1\end{bmatrix}t$$

Since any linear multiple of an eigenvector is also an eigenvector, any t will do. Choosing $t=-3$:

$$\tilde{v}_1=\begin{bmatrix}0\\2\\-3\end{bmatrix},\lambda=2$$

Using exactly the same methods, we can find the corresponding eigenvectors for $\lambda=3$ (this time there will be two because of the degeneracy):

$$A(t)\tilde{v} = \lambda\tilde{v}$$
$$(A(t) - \lambda I)\tilde{v} = \tilde{0}$$
$$\left(\begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & -2 \\ 3 & 6 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)\tilde{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & -2 \\ 3 & 6 & 3 \end{bmatrix}\tilde{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Write in augmented matrix form:

$$\begin{bmatrix} 0 & 0 & 0 & | & 0 \\ 2 & -4 & -2 & | & 0 \\ 3 & 6 & 3 & | & 0 \end{bmatrix}$$

Solving using row reduction (steps omitted):

$$\begin{bmatrix} 0 & 0 & 0 & | & 0 \\ 1 & 0 & 0 & | & 0 \\ 0 & 2 & 1 & | & 0 \end{bmatrix}$$

Thus:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence $v_1 = 0$, $v_3 = t$, and v_2:

$$2v_2 + v_3 = 0$$
$$v_2 = -\frac{t}{2}$$
$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix} t$$

Since any linear multiple of an eigenvector is also an eigenvector, any t will do. Choosing $t = -2$:

$$\tilde{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \lambda = 3$$

These are the only two eigenvectors of the matrix A, but we still need a third solution. To get this last solution, we use the degeneracy of order 2 of $\lambda = 3$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & -2 \\ 3 & 6 & 3 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & -2 \\ 3 & 6 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & -2 \\ 3 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -14 & 4 & 2 \\ 21 & -6 & -3 \end{bmatrix}$$

Now using row reduction:

$$\begin{bmatrix} 0 & 0 & 0 & | & 0 \\ -14 & 4 & 2 & | & 0 \\ 21 & -6 & -3 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 7 & -2 & -1 & | & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 7 & -2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence $v_1 = t, v_2 = s, v_3$:

$$7v_1 - 2v_2 - v_3 = 0$$
$$7t - 2s = v_3$$
$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} s + \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} t$$

Since any linear multiple of an eigenvector is also an eigenvector, any two combinations of s and t will do. Choosing $(s, t) = (1,0)$ and $(3,1)$:

$$\tilde{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Note that the first solution here is just the same as $\tilde{v}_2$, which is as it should be, but obviously we don't need this same solution twice. So just keep:

$$\tilde{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \lambda = 3$$

Note that $\tilde{v}_3$is not actually an eigenvector, but it is what we call a generalised eigenvector. We need it for constructing the final set of solutions.

Now that we have our three generalised eigenvectors and corresponding eigenvalues, we can go way back to the beginning and remember that solutions to the homogenous equation will be of the form:

$$\tilde{x}(t) = e^{\lambda t}\tilde{v}$$

Hence we have three linearly independent solutions to the homogenous equation:

$$\tilde{x}_1(t) = e^{\lambda t}\tilde{v}_1 = e^{2t}\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$
$$\tilde{x}_2(t) = e^{\lambda t}\tilde{v}_2 = e^{3t}\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$
$$\tilde{x}_3(t) = e^{\lambda t}\tilde{v}_3 + te^{\lambda t}\begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & -2 \\ 3 & 6 & 3 \end{bmatrix}\tilde{v}_3$$
$$= e^{3t}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + te^{3t}\begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & -2 \\ 3 & 6 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$
$$= e^{3t}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + te^{3t}\begin{bmatrix} 0 \\ -12 \\ 24 \end{bmatrix}$$
$$\tilde{x}_3(t) = e^{3t}\begin{bmatrix} 1 \\ 3 - 12t \\ 1 + 24t \end{bmatrix}$$

Note that the solution corresponding to the generalised eigenvector is more complicated than the rest (which is why we don't like repeated eigenvalues). The general formula for this is:

$$\tilde{x}(t) = e^{\lambda t} \sum_{k=0}^{l-1} \frac{1}{k!} t^k (A - \lambda I)^k \tilde{v}$$

Where λ is the eigenvalue corresponding to the generalised eigenvector $\tilde{v}$, and l is the degeneracy of the repeated eigenvalue.

We now write each of the independent solutions as the column of a matrix called $X(t)$:

$$X(t) = \begin{bmatrix} 0 & 0 & e^{3t} \\ 2e^{2t} & e^{3t} & (3-12t)e^{3t} \\ -3e^{2t} & -2e^{3t} & (1+24t)e^{3t} \end{bmatrix}$$

Next, we define a very important matrix called the fundamental matrix $M(t|t_0)$:

$$M(t|t_0) = X(t)X^{-1}(t_0)$$

Note that in order to avoid division by zero, it is generally best when finding M to evaluate $X(t_0)$ at $t = 0$ first before finding its inverse. In this instance we have:

$$M(t|t_0) = X(t)X^{-1}(t_0)$$
$$M(t|t_0) = \begin{bmatrix} 0 & 0 & e^{3t} \\ 2e^{2t} & e^{3t} & (3-12t)e^{3t} \\ -3e^{2t} & -2e^{3t} & (1+24t)e^{3t} \end{bmatrix} \begin{bmatrix} -7 & 2 & 1 \\ 11 & -3 & -2 \\ 1 & 0 & 0 \end{bmatrix}$$
$$M(t|t_0) = \begin{bmatrix} e^{3t} & 0 & 0 \\ -14e^{2t} + (14-12t)e^{3t} & 4e^{2t} - 3e^{3t} & 2e^{2t} - 2e^{3t} \\ 21e^{2t} - (21-24t)e^{3t} & -6e^{2t} + 6e^{3t} & -3e^{2t} + 4e^{3t} \end{bmatrix}$$

Since it is comprised only of intependent solutions of the homogenous equation, the fundamental matrix itself also satisfies the equations:

$$M'(t|t_0) = A(t)M(t|t_0)$$
$$M(t_0|t_0) = I$$

Now that we have $M(t|t_0)$, we can use a very similar trick to that employed in order to solve a second-order inhomogenous linear ODE. Namely, write:

$$\tilde{x}(t) = M(t|t_0)\tilde{s}(t)$$
$$\tilde{x}'(t) = M'(t|t_0)\tilde{s}(t) + M(t|t_0)\tilde{s}'(t)$$

Substituting this into the original ODE:

$$\tilde{x}'(t) = A(t)\tilde{x} + f(t)$$
$$M'(t|t_0)\tilde{s}(t) + M(t|t_0)\tilde{s}'(t) = A(t)M(t|t_0)\tilde{s}(t) + f(t)$$
$$A(t)M(t|t_0)\tilde{s}(t) + M(t|t_0)\tilde{s}'(t) = A(t)M(t|t_0)\tilde{s}(t) + f(t)$$
$$M(t|t_0)\tilde{s}'(t) = f(t)$$
$$\tilde{s}'(t) = M(t|t_0)^{-1}f(t)$$
$$\tilde{s}(t) = \tilde{s}(t_0) + \int_{t_0}^{t} M(\tau|t_0)^{-1} f(\tau)\, d\tau$$
$$M(t|t_0)^{-1}\tilde{x}(t) = \tilde{s}(t_0) + \int_{t_0}^{t} M(\tau|t_0)^{-1} f(\tau)\, d\tau$$

$$\tilde{x}(t) = M(t|t_0)\tilde{s}(t_0) + M(t|t_0)\int_{t_0}^{t} M(\tau|t_0)^{-1} f(\tau)\, d\tau$$

Note that:

$$\tilde{x}(t_0) = M(t_0|t_0)\tilde{s}(t_0) = \tilde{s}(t_0)$$

Hence:

$$\tilde{x}(t) = M(t|t_0)\tilde{x}(t_0) + M(t|t_0)\int_{t_0}^{t} M(\tau|t_0)^{-1} f(\tau)\, d\tau$$

$$\tilde{x}(t) = M(t|t_0)\tilde{x}(t_0) + \int_{t_0}^{t} M(t|t_0)M(\tau|t_0)^{-1} f(\tau)\, d\tau$$

$$\tilde{x}(t) = M(t|t_0)\tilde{x}(t_0) + \int_{t_0}^{t} M(t-\tau|t_0) f(\tau)\, d\tau$$

For this particular example, we make the relevant substitutions, and using $\tilde{x}(t_0) = (1,0,0)$:

$$\tilde{x}(t) = \begin{bmatrix} e^{3t} & 0 & 0 \\ -14e^{2t} + (14-12t)e^{3t} & 4e^{2t} - 3e^{3t} & 2e^{2t} - 2e^{3t} \\ 21e^{2t} - (21-24t)e^{3t} & -6e^{2t} + 6e^{3t} & -3e^{2t} + 4e^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$+ \int_0^t \begin{bmatrix} e^{3(t-\tau)} & 0 & 0 \\ -14e^{2(t-\tau)} + \left(14 - 12(t-\tau)\right)e^{3(t-\tau)} & 4e^{2(t-\tau)} - 3e^{3(t-\tau)} & 2e^{2(t-\tau)} - 2e^{3(t-\tau)} \\ 21e^{2(t-\tau)} - \left(21 - 24(t-\tau)\right)e^{3(t-\tau)} & -6e^{2(t-\tau)} + 6e^{3(t-\tau)} & -3e^{2(t-\tau)} + 4e^{3(t-\tau)} \end{bmatrix} \begin{bmatrix} e^{3\tau} \\ 0 \\ 0 \end{bmatrix} d\tau$$

$$\tilde{x}(t) = \begin{bmatrix} e^{3t} & 0 & 0 \\ -14e^{2t} + (14-12t)e^{3t} & 4e^{2t} - 3e^{3t} & 2e^{2t} - 2e^{3t} \\ 21e^{2t} - (21-24t)e^{3t} & -6e^{2t} + 6e^{3t} & -3e^{2t} + 4e^{3t} \end{bmatrix}$$

$$+ \int_0^t \begin{bmatrix} e^{3t} \\ -14e^{2(t-\tau)} + \left(14 - 12(t-\tau)\right)e^{3(t-\tau)} \\ 21e^{2(t-\tau)} - \left(21 - 24(t-\tau)\right)e^{3(t-\tau)} \end{bmatrix} d\tau$$

$$\tilde{x}(t) = \begin{bmatrix} (1+t)e^{3t} \\ (2t - 6t^2)e^{3t} \\ (3t + 12t^2)e^{3t} \end{bmatrix}$$

Integral Transform Methods

Important Properties of Fourier Series

- The Fourier series of a function can always be defined, but it will not necessarily converge
- If a function satisfies Dirichlet's conditions (bounded, finitely many points of discontinuity, finitely many local maxima and minima) on a finite interval, then at any interior point the Fourier series converges to the value of the function at that point
- Fejer's Theorem: no two different continuous functions can have the same set of Fourier coefficients
- If we are given a non-periodic function to represent as a Fourier series, we can take either the odd or even extension of the function
- A uniformly convergent Fourier series can be integrated term-by-term. Term-by-term differentiation is determined by the quality of convergence of the differentiated series, not the original series

Important Properties of the Fourier Transform

- Absolute convergence condition: If $f(t)$ is Riemann integrable, with the integral $\int_{-\infty}^{\infty}|f(t)|\,dt < \infty$ on all finite intervals, then $\tilde{f}(q)$ exists for all $q \in \mathbb{R}$
- Conditional convergence condition: If $f(t)$ is Riemann integrable, $f(t) \to 0$ as $t \to \infty$, and for some $L > 0$ $f(t)$ is monotonic, then $\tilde{f}(q)$ exists for all nonzero $q \in \mathbb{R}$
- Fourier transforms are defined from positive to negative infinity, so they are particularly useful when the domain of our function is unbounded
- When we solve an ODE by taking the Fourier transform of both sides, we are implicitly assuming that the Fourier transform exists for the unknown function. We need to check that this is the case after we found our solution
- The Fourier cosine transform is used to extend a function defined only over a positive domain across the entire reals. It is defined as: $f(x) = \frac{2}{\pi}\int_0^{\infty}\cos(\xi x)\, f_c(\xi)\, d\xi$
- The Fourier sine transform is similar to the cosine transform except that it produces an odd rather than an even function. It is defined as: $f(x) = \frac{2}{\pi}\int_0^{\infty}\sin(\xi x)\, f_s(\xi) d\xi$
- In solving ODEs, the sine transform is useful if we have a Dirichlet boundary condition, while the cosine transform is useful for Neumann boundary conditions

Important Properties of the Laplace Transform

- If $f(t)$ is Riemann integrable, with the integral $\int_0^{\infty}|f(t)|\,dt < \infty$ on all finite intervals and $f(t) \leq Ke^{\sigma t}\ as\ t \to \infty$, then $\hat{f}(s)$ exists for $s > \sigma$
- Lerch's Theorem states that if $f(t)$ and $g(t)$ are continuous on $(0, \infty)$ and have the same laplace transform, then $f(t) = g(t)$
- Laplace transforms are particularly useful for initial value problems, as taking Laplace transforms of derivatives naturally introduces initial values in the problem
- When we solve an ODE by taking the Laplace transform of both sides, we are implicitly assuming that the Laplace transform exists for the unknown function. We need to check that this is the case after we found our solution

Proof of Laplace Transform Convolution Theorem

$$
\begin{aligned}
L[(f*g)(t); t \to s] &= \int_0^{\infty} e^{-st}\int_0^{t} f(\tau)g(t-\tau)\,d\tau\,dt \\
&= \int_0^{\infty} f(\tau)\int_{\tau}^{\infty} e^{-st}g(t-\tau)\,dtd\tau \\
&= \int_0^{\infty} e^{-s\tau}f(\tau)\int_{\tau}^{\infty} e^{-st}e^{s\tau}g(t-\tau)\,dtd\tau \\
&= \int_0^{\infty} e^{-s\tau}f(\tau)\int_{\tau}^{\infty} e^{-s(t-\tau)}g(t-\tau)\,dtd\tau \\
\text{let } t &= u+\tau \\
&= \int_0^{\infty} e^{-s\tau}f(\tau)\int_0^{\infty} e^{-su}g(u)\,dud\tau \\
&= \int_0^{\infty} e^{-s\tau}f(\tau)\,\hat{g}(s)d\tau \\
&= \hat{g}(s)\int_0^{\infty} e^{-s\tau}f(\tau)\,d\tau \\
L[(f*g)(t); t \to s] &= \hat{g}(s)\hat{f}(s)
\end{aligned}
$$

Proof of s-Shift Theorem

$$L[H(t-T)f(t-T);t\to s]=\int_0^\infty e^{-st}\,H(t-T)f(t-T)dt$$
$$=\int_T^\infty e^{-st}\,f(t-T)dt$$
let $t=T+\tau$
$$=\int_0^\infty e^{-s(T+\tau)}\,f(\tau)d\tau$$
$$=e^{-sT}\int_0^\infty e^{-s\tau}\,f(\tau)d\tau$$
$$L[H(t-T)f(t-T);t\to s]=e^{-sT}\hat{f}(s)$$

Proof of Periodic Function Transform

$$\hat{f}(s)=\int_0^\infty e^{-st}f(t)\,dt$$
$$=\sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} e^{-st}f(t)\,dt$$
let $t=nT+\tau$
$$=\sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} e^{-st}f(t)\,dt$$
$$\hat{f}(s)=\sum_{n=0}^{\infty}\int_0^T e^{-s(nT+\tau)}f(nT+\tau)\,d\tau$$

By the property of periodicity $f(t+T)=f(t)$, hence:

$$\hat{f}(s)=\sum_{n=0}^{\infty}\int_0^T e^{-s(nT+\tau)}f(\tau)\,d\tau$$
$$=\sum_{n=0}^{\infty}e^{-snT}\int_0^T e^{-s\tau}f(\tau)\,d\tau$$
$$\hat{f}(s)=\sum_{n=0}^{\infty}[e^{-sT}]^n\int_0^T e^{-s\tau}f(\tau)\,d\tau$$

Identify the geometric series:

$$\hat{f}(s)=\frac{1}{1-e^{-sT}}\int_0^T e^{-s\tau}f(\tau)\,d\tau$$

Green's Function

The Green function is used for solving differential equations with source terms and homogeneous boundary conditions. It is defined as the function that satisfies the following property (for any linear differential operator L):

$$Lu(x)=f(x)$$
$$u(x)=\int G(x,x')f(x')\,dx'$$

Thus, the Green function allows us to express the solution to a PDE in terms of an integral. The trick is finding the Green function for a given equation and set of boundary conditions. There are two main ways of going about this: by an integral transform, and by direct calculation.

Consider the example of solving $y''(x) - k^2 y(x) = -f(x)$ where $y(0) = y_0$:

Take the Fourier sine transform:

$$F_s[y''(x); x \to \xi] = \xi y_0 - \xi^2 \tilde{y}_s(\xi) - k^2 \tilde{y}_s(\xi) = -\tilde{f}_s(\xi)$$

$$\tilde{y}_s(\xi)[\xi^2 + k^2] = \xi y_0 + \tilde{f}_s(\xi)$$

$$\tilde{y}_s(\xi) = \frac{\xi y_0 + \tilde{f}_s(\xi)}{\xi^2 + k^2}$$

$$\tilde{y}_s(\xi) = \frac{\xi y_0}{\xi^2 + k^2} + \frac{\tilde{f}_s(\xi)}{\xi^2 + k^2}$$

Invert the transform:

$$y(x) = \frac{2}{\pi}\int_0^\infty \frac{\xi y_0}{\xi^2 + k^2} + \frac{\tilde{f}_s(\xi)}{\xi^2 + k^2} d\xi$$

$$y(x) = \frac{2}{\pi}\int_0^\infty \frac{\xi y_0}{\xi^2 + k^2} d\xi + \frac{2}{\pi}\int_0^\infty \frac{\tilde{f}_s(\xi)}{\xi^2 + k^2} d\xi$$

Evaluating the first integral using tables, and writing the transform out explicitly in the second:

$$y(x) = y_0 e^{-kx} + \frac{2}{\pi}\int_0^\infty \frac{1}{\xi^2 + k^2}\int_0^\infty f(x')\sin(\xi x')\, dx' d\xi$$

$$= y_0 e^{-kx} + \frac{2}{\pi}\int_0^\infty \int_0^\infty \frac{1}{\xi^2 + k^2} f(x')\sin(\xi x')\, d\xi dx'$$

$$= y_0 e^{-kx} + \frac{2}{\pi}\int_0^\infty f(x')\int_0^\infty \frac{1}{\xi^2 + k^2}\sin(\xi x')\, d\xi dx'$$

$$y(x) = y_0 e^{-kx} + \frac{2}{\pi}\int_0^\infty f(x') G(x', \xi)\, dx'$$

In evaluating the Green Function directly, the first step is to find a homogenous equation for which the Green Function is the solution. One then solves this for the Green Function, using boundary values and continuity properties in order to determine all unknown coefficients. Consider for example the equation:

$$\frac{\partial^2 G(x)}{\partial x^2} = -\delta(x - \xi)$$

Where $u'(0) = 0, u(1) = 0$

Solving this for $G(x)$ yields:

$$G(x) = \begin{cases} -Ax + B, & 0 < x < \xi \\ -Cx + D, & \xi < x < 1 \end{cases}$$

From neumann boundary condition:

$$G'(0): -A = 0 \Rightarrow A = 0$$

From dirichlet boundary condition:

$$G(1): -C + D = 0 \Rightarrow C = -D$$

From continuity condition:

$$-A\xi + B = -C\xi + D$$

$$B = -C\xi - C$$

From derivative jump condition:

$$-A - (-C) = -1$$
$$C = -1$$

Hence:

$$D = -(-1) = 1$$
$$B = -C\xi - C$$
$$B = \xi + 1$$

Thus we have:

$$G(x) = \begin{cases} \xi + 1, & 0 < x < \xi \\ x + 1, & \xi < x < 1 \end{cases}$$

Partial Differential Equations

Minimum and Maximum Principle for Harmonic Functions

A harmonic function is any function that satisfies the Laplace equation: that is it has a second derivative of zero everywhere. The minimum and maximum principles state that harmonic functions always attain their minimum and maximum values on the boundary. This is a simple consequence of the nature of harmonic functions: if they had any internal turning points, their second derivative would not be zero, and hence they would not be harmonic functions.

D'Alembert's Solution to the Wave Equation

This is a simple method for solving the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

The key of this method is the make the following change of variables:

$$\xi = x - ct, \qquad \eta = x + ct, \qquad u(x,t) = v(\xi, \eta)$$

Hence:

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial t}$$
$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial \xi}(-c) + \frac{\partial v}{\partial \eta}(c)$$
$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial \eta} vc - \frac{\partial}{\partial \xi} vc$$
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial \eta} c \frac{\partial v}{\partial t} - \frac{\partial}{\partial \xi} c \frac{\partial v}{\partial t}$$

Substituting in our earlier expression:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial \eta} c \left(\frac{\partial}{\partial \eta} vc - \frac{\partial}{\partial \xi} vc\right) - \frac{\partial}{\partial \xi} c \left(\frac{\partial}{\partial \eta} vc - \frac{\partial}{\partial \xi} vc\right)$$
$$= c^2 \left(\frac{\partial^2 v}{\partial \eta^2} - \frac{\partial^2 v}{\partial \xi \partial \eta}\right) - c^2 \left(\frac{\partial^2 v}{\partial \eta \partial \xi} - \frac{\partial^2 v}{\partial \xi \partial \eta}\right)$$
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial \eta^2} - 2c^2 \frac{\partial^2 v}{\partial \xi \partial \eta} + c^2 \frac{\partial^2 v}{\partial \xi^2}$$

By symmetry the derivative with respect to x will yield:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}$$

Hence we can substitute these expressions into the original form of the equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$c^2 \frac{\partial^2 v}{\partial \eta^2} - 2c^2 \frac{\partial^2 v}{\partial \xi \partial \eta} + c^2 \frac{\partial^2 v}{\partial \xi^2} = c^2 \left(\frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2} \right)$$

$$-\frac{\partial^2 v}{\partial \xi \partial \eta} = \frac{\partial^2 v}{\partial \xi \partial \eta}$$

The only valid solution for this is:

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = 0$$

Integrating one with respect to each variable we find:

$$v(\xi, \eta) = f(\xi) + g(\eta)$$

$$u(x,t) = f(x - ct) + g(x + ct)$$

Given initial conditions $u(x,0) = u_0(x)$ and $x_t(x,0) = v_0(x)$, take the derivative:

$$u_t(x,t) = f'(x - ct)(-c) + g'(x + ct)(c)$$

$$u_t(x,t) = c[g'(x + ct) - f'(x - ct)]$$

Evaluate derivative at the initial condition and integrate:

$$v_0(x) = c[g'(x) - f'(x)]$$

$$\int_0^x \frac{1}{c} v_0(s)\, ds = g(x) - f(x)$$

Evaluate original equation at initial condition:

$$u(x,t) = f(x - ct) + g(x + ct)$$

$$u(x) = f(x) + g(x)$$

Solve as simultaneous equations:

$$\int_0^x \frac{1}{c} v_0(s)\, ds + f(x) = g(x)$$

$$\int_0^x \frac{1}{c} v_0(s)\, ds + u(x) - g(x) = g(x)$$

$$g(x) = \frac{1}{2} \int_0^x \frac{1}{c} v_0(s)\, ds + \frac{1}{2} u(x)$$

$$f(x) = u(x) - g(x)$$

$$f(x) = u(x) - \frac{1}{2} \int_0^x \frac{1}{c} v_0(s)\, ds - \frac{1}{2} u(x)$$

$$f(x) = \frac{1}{2} \int_0^x \frac{1}{c} v_0(s)\, ds + \frac{1}{2} u(x)$$

Substitute back into original solution:

$$u(x,t) = f(x-ct) + g(x+ct)$$
$$= \frac{1}{2}\int_0^x \frac{1}{c} v_0(s)\,ds + \frac{1}{2}u(x) + \frac{1}{2}\int_0^x \frac{1}{c} v_0(s)\,ds + \frac{1}{2}u(x)$$
$$= \frac{1}{2c}\int_0^{x-ct} v_0(s)\,ds + \frac{1}{2}u(x-ct) + \frac{1}{2c}\int_0^{x+ct} v_0(s)\,ds + \frac{1}{2}u(x+ct)$$
$$u(x,t) = \frac{1}{2}[u(x-ct) + u(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(s)\,ds$$

Diffusion Equation with Steady-State

Consider the diffusion equation:

$$T_t = DT_{xx}$$
$$T(0,t) = T_0, T(L,t) = T_1, T(x,0) = f(x)$$

In order to solve this inhomogeneous equation, we use the trick of writing the solution as the sum of a steady-state $g(x)$ and a transient solution $v(x,t)$:

$$u(x,t) = g(x) + v(x,t)$$

First consider the transient portion. We seek a separated solution of the form:

$$v(t,x) = T(t)X(x)$$

Hence:

$$T'(t)X(x) = DT(t)X''(x)$$
$$\frac{1}{D}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

Thus we separate the equation into two ODEs:

$$\frac{1}{D}\frac{T'(t)}{T(t)} = -\lambda$$
$$T'(t) = -\lambda DT(t)$$
$$\frac{1}{T(t)}\frac{dT(t)}{dt} = -\lambda D$$
$$\int \frac{1}{T(t)}\,dT(t) = \int -\lambda D\,dt$$
$$\log T(t) = -\lambda Dt + \mathrm{C}$$
$$T(t) = e^{-\lambda D + \mathrm{C}}$$
$$T(t) = Ce^{-\lambda Dt}$$

$$\frac{X''(x)}{X(x)} = -\lambda$$
$$X''(x) + \lambda X(x) = 0$$

Try solution of the form

$$X(x) = A\cos(\lambda^{1/2}x) + B\sin(\lambda^{1/2}x)$$

From the homogenous boundary condition (Dirichlet):

$$0 = A\cos(0) + B\sin(0)$$
$$\therefore A = 0$$

$$0 = B\sin\left(\lambda^{1/2}L\right)$$
$$\therefore \lambda^{1/2}L = n\pi$$
$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Putting together the parts we find:

$$T(t,x) = T(t)X(x)$$
$$= Ce^{-\lambda Dt}\left[A\cos\left(\lambda^{1/2}x\right) + B\sin\left(\lambda^{1/2}x\right)\right]$$
$$T(t,x) = B\sin\left(\frac{n\pi}{L}x\right)e^{-\lambda Dt}$$

Using the principle of superposition (i.e. that the linear combination of any solutions is also a solution):

$$T(t,x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)e^{-\lambda Dt}$$

Now we consider the steady-state solution, which will have the form:

$$T_{xx} = 0$$
$$t_x = C$$
$$t = C + Bx$$

$$g(x) = T_0 + (T_1 - T_0)x$$

Using this result, we can re-write the initial value for the homogenous problem:

$$u(t,x) = g(x) + v(x,t)$$
$$u(0,x) = T_0 + (T_1 - T_0)x + v(x,0)$$
$$f(x) = T_0 + (T_1 - T_0)x + v(x,0)$$
$$v(0,x) = f(x) - T_0 - (T_1 - T_0)x$$

Now that we have a homogenous problem (with the proper adjustment made to the initial value), we can simply substitute in the initial value to find a B_n:

$$T(t,x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)e^{-\lambda Dt}$$
$$T(0,x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$
$$f(x) - T_0 - (T_1 - T_0)x = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

We can solve for B_n using orthogonality:

$$[f(x) - T_0 - (T_1 - T_0)x]\sin\left(\frac{m\pi}{L}x\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)$$

$$\int_0^L [f(x) - T_0 - (T_1 - T_0)x] \sin\left(\frac{m\pi}{L}x\right) dx = \int_0^L \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx$$
$$\int_0^L [f(x) - T_0 - (T_1 - T_0)x] \sin\left(\frac{m\pi}{L}x\right) dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx$$

By orthogonality, the left most integral is zero unless $m = n$:

$$\int_0^L [f(x) - T_0 - (T_1 - T_0)x] \sin\left(\frac{m\pi}{L}x\right) dx = \frac{L}{2} B_n$$
$$B_n = \frac{2}{L} \int_0^L [f(x) - T_0 - (T_1 - T_0)x] \sin\left(\frac{m\pi}{L}x\right) dx$$

Hence we have our complete solution:

$$T(t,x) = \sum_{n=1}^{\infty} \frac{2}{L} \left[\int_0^L [f(x) - T_0 - (T_1 - T_0)x] \sin\left(\frac{n\pi}{L}x\right) dx \right] \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda D t}$$

Poisson Equation with Symmetric Solution

Consider the inhomogeneous Poisson equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -1$$
$$u(x,y) = x + y \text{ on the boundary}$$

In order to solve this, we decompose the solution into radially symmetric and non-radially symmetric components:

$$u(x,y) = Q(r,\theta) + P(r)$$

First consider the solution for the homogenous equation $U(x,y)$. Begin by converting to cylindrical coordinates:

$$\frac{\partial^2 Q}{\partial r^2} + \frac{1}{r}\frac{\partial Q}{\partial r} + \frac{1}{r^2}\frac{\partial^2 Q}{\partial \theta^2} = 0$$

As usual we seek a separated solution of the form:

$$Q(r,\theta) = \Theta(\theta)R(r)$$
$$Q'(r,\theta) = \Theta'(\theta)R(r) + \Theta(\theta)R'(r)$$
$$Q''(r,\theta) = \Theta''(\theta)R(r) + 2\Theta'(\theta)R'(r) + \Theta(\theta)R''(r)$$

Substituting this into the PDE:

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$
$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) = -\frac{1}{r^2}R(r)\Theta''(\theta)$$
$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = -\lambda$$

Consider first the ODE for the radial term:

$$r^2\frac{R''(r)}{R(r)}+r\frac{R'(r)}{R(r)}=-\lambda$$
$$r^2R''(r)+rR'(r)=-\lambda R(r)$$
$$r^2R''(r)+rR'(r)+\lambda R(r)=0$$

This is a Cauchy-Euler equation, which has solutions of the form r^n. Now considering the second ODE:

$$-\frac{\Theta''(\theta)}{\Theta(\theta)}=-\lambda$$
$$\Theta''(\theta)+\lambda\Theta(\theta)=0$$

This has solutions of the form:

$$\Theta(\theta)=A\cos\left(\lambda^{1/2}\theta\right)+B\sin\left(\lambda^{1/2}\theta\right)$$

From the boundary condition, $\Theta(\pi)=\Theta(-\pi)$, hence it must follow that:

$$\lambda^{1/2}L=2\pi n$$

Since $L=2\pi$:

$$\lambda=n^2$$
$$\Theta(\theta)=A\cos(n\theta)+B\sin(n\theta)$$

Combine these separated solutions in the original homogenous equation:

$$R''(r)\Theta(\theta)+\frac{1}{r}R'(r)\Theta(\theta)+\frac{1}{r^2}R(r)\Theta''(\theta)=0$$
$$u(r,\theta)=\Theta(\theta)R(r)$$
$$Q(r,\theta)=[A\cos(n\theta)+B\sin(n\theta)]r^n$$

The full set of solutions (by the superposition theorem) is given by:

$$Q(r,\theta)=\frac{A_0}{2}+\sum_{n=1}^{\infty}[A_n\cos(n\theta)+B_n\sin(n\theta)]r^n$$

Substituting in our boundary condition:

$$Q(r,\theta)=\frac{A_0}{2}+\sum_{n=1}^{\infty}[A_n\cos(n\theta)+B_n\sin(n\theta)]r^n$$
$$\cos\theta+\sin\theta=\frac{A_0}{2}+\sum_{n=1}^{\infty}[A_n\cos(n\theta)+B_n\sin(n\theta)]r^n$$

From orthogonality we need $A_1=B_1=1$ and $A_n=B_n=0$ for $n\neq 1$:

$$Q(r,\theta)=r\cos\theta+r\sin\theta$$

Now that we have the solution for the homogenous part, we need to find the solution for the radially symmetric part $P(x,y)$. The symmetry means this can be written in the form:

$$P''(r)+\frac{1}{r}P'(r)=-1$$
$$\frac{1}{r}\frac{d}{dr}\left(rP'(r)\right)=-1$$

$$\int d\big(rP'(r)\big) = -\int r\,dr$$
$$rP'(r) = -\frac{1}{2}r^2 + A$$
$$P'(r) = -\frac{1}{2}r + \frac{A}{r}$$
$$\frac{dP(r)}{dr} = -\frac{1}{2}r + \frac{A}{r}$$
$$\int dP(r) = \int -\frac{1}{2}r + \frac{A}{r}dr$$
$$P(r) = -\frac{1}{4}r^2 + A\log r + B$$

To avoid blowing up as $r \to 0$, drop the log term:

$$P(r) = -\frac{1}{4}r^2 + B$$

Substituting $P(1) = 0$:

$$0 = -\frac{1}{4} + B$$
$$B = \frac{1}{4}$$

Hence:

$$P(r) = -\frac{1}{4}r^2 + \frac{1}{4}$$
$$P(r) = \frac{1-r^2}{4}$$

Our complete solution is simply the sum of the two parts:

$$u(r,\theta) = P(r) + Q(r,\theta) = \frac{1-r^2}{4} + r\cos\theta + r\sin\theta$$

Laplace Equation on a Rectangle

Consider the Laplace equation:

$$u_{xx} + u_{yy} = 0$$

Subject to the boundary conditions:

$$u(0,y) = p(y), 0 < y < b$$
$$u(a,y) = q(y), 0 < y < b$$
$$u(x,0) = r(x), 0 < x < a$$
$$u(x,b) = s(x), 0 < x < a$$

It is convenient in this instance to consider a solution separated into two parts:

$$u(x,y) = U(x,y) + V(x,y)$$

$$\begin{array}{lll} U(0,y) = 0, & V(0,y) = p(y), & 0 < y < b \\ U(a,y) = 0, & V(a,y) = q(y), & 0 < y < b \\ U(x,0) = r(x), & V(x,0) = 0, & 0 < x < a \\ U(x,b) = s(x), & V(x,b) = 0, & 0 < x < a \end{array}$$

First we solve the function $U(x,y)$ by looking for separated solutions:

$$U(x,y) = X(x)Y(x)$$

Taking the $X(x)$ term first, it has solutions of the form:

$$X''(x) + \lambda X(x) = 0, \qquad X(0) = X(a) = 0$$

Solutions are of the form:

$$X(x) = A\cos\left(\frac{n\pi x}{L}\right) + B\sin\left(\frac{n\pi x}{L}\right)$$

Substituting in the boundary conditions we have:

$$X(x) = B\sin\left(\frac{n\pi x}{a}\right)$$

The eigenvalue is thus given by:

$$\lambda = \frac{n^2\pi^2}{a^2}$$

Now turning to $Y(y)$, this has solutions of the form:

$$Y''(y) - \lambda Y(y) = 0, \qquad Y(0) = r(x), Y(b) = s(x)$$

Solutions are of the form:

$$Y(y) = B\cosh\left(\frac{n\pi y}{a}\right) + C\sinh\left(\frac{n\pi y}{a}\right)$$

Putting the full solution together for $U(x,y)$:

$$U(x,y) = B\sin\left(\frac{n\pi x}{a}\right)\left[B\cosh\left(\frac{n\pi y}{a}\right) + C\sinh\left(\frac{n\pi y}{a}\right)\right]$$

$$U(x,y) = \sum_{n=1}^{\infty}\left[A_n\cosh\left(\frac{n\pi y}{a}\right) + B_n\sinh\left(\frac{n\pi y}{a}\right)\right]\sin\left(\frac{n\pi x}{a}\right)$$

Substituting in the boundary conditions we have:

$$\sum_{n=1}^{\infty}[A_n\cosh(0) + B_n\sinh(0)]\sin\left(\frac{n\pi x}{a}\right) = r(x)$$

$$\sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi x}{a}\right) = r(x)$$

$$\sum_{n=1}^{\infty}\left[A_n\cosh\left(\frac{n\pi b}{a}\right) + B_n\sinh\left(\frac{n\pi b}{a}\right)\right]\sin\left(\frac{n\pi x}{a}\right) = r(x)$$

$$\sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi x}{a}\right) = r(x)$$

We can then use orthogonality to solve for the coefficients:

$$\sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right) = \sin\left(\frac{n\pi x}{a}\right)r(x)$$

$$\sum_{n=1}^{\infty}\left[A_n\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)dx\right] = \int_0^a \sin\left(\frac{n\pi x}{a}\right)r(x)\,dx$$

$$A_n \frac{a}{2} = \int_0^a r(x) \sin\left(\frac{n\pi x}{a}\right) dx$$
$$A_n = \frac{2}{a} \int_0^a r(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

$$\sum_{n=1}^{\infty} \left[A_n \cosh\left(\frac{n\pi b}{a}\right) + B_n \sinh\left(\frac{n\pi b}{a}\right)\right] \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) = r(x) \sin\left(\frac{n\pi x}{a}\right)$$
$$\sum_{n=1}^{\infty} \int_0^a \left[A_n \cosh\left(\frac{n\pi b}{a}\right) + B_n \sinh\left(\frac{n\pi b}{a}\right)\right] \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx = \int_0^a r(x) \sin\left(\frac{n\pi x}{a}\right) dx$$
$$\frac{a}{2}\left[A_n \cosh\left(\frac{n\pi b}{a}\right) + B_n \sinh\left(\frac{n\pi b}{a}\right)\right] = \int_0^a r(x) \sin\left(\frac{n\pi x}{a}\right) dx$$
$$B_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a} \int_0^a r(x) \sin\left(\frac{n\pi x}{a}\right) dx - \frac{2}{a} \cosh\left(\frac{n\pi b}{a}\right) \left[\int_0^a r(x) \sin\left(\frac{n\pi x}{a}\right) dx\right]$$
$$B_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a} \left[\int_0^a r(x) \sin\left(\frac{n\pi x}{a}\right) dx\right] \left(1 - \cosh\left(\frac{n\pi b}{a}\right)\right)$$
$$B_n = \frac{2}{a} \left[\int_0^a r(x) \sin\left(\frac{n\pi x}{a}\right) dx\right] \left(\frac{1 - \cosh\left(\frac{n\pi b}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\right)$$

The method for the other half of this problem is exactly symmetrical. The complete solution should look something like this:

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L} y\right) \frac{\sinh\left(\frac{n\pi}{L}(K-x)\right)}{\sinh\left(\frac{n\pi}{L} K\right)} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} y\right) \frac{\sinh\left(\frac{n\pi}{L} x\right)}{\sinh\left(\frac{n\pi}{L} K\right)}$$
$$+ \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{K} x\right) \frac{\sinh\left(\frac{n\pi}{K}(L-y)\right)}{\sinh\left(\frac{n\pi}{K} L\right)} + \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi}{K} x\right) \frac{\sinh\left(\frac{n\pi}{K} y\right)}{\sinh\left(\frac{n\pi}{K} L\right)}$$

coefficients a_n, b_n, c_n, d_n calculated by

$$a_n = \frac{2}{L} \int_0^L g_1(y) \sin\left(\frac{n\pi}{L} y\right) dy, \qquad n \in \mathbb{N},$$
$$b_n = \frac{2}{L} \int_0^L g_2(y) \sin\left(\frac{n\pi}{L} y\right) dy, \qquad n \in \mathbb{N},$$
$$c_n = \frac{2}{K} \int_0^K f_1(x) \sin\left(\frac{n\pi}{K} x\right) dx, \qquad n \in \mathbb{N},$$
$$d_n = \frac{2}{K} \int_0^K f_2(x) \sin\left(\frac{n\pi}{K} x\right) dx, \qquad n \in \mathbb{N}.$$

Proof that Eigenvalues are Positive

To prove that in standard homogeneous boundary conditions the eigenvalues are always positive, begin with the differential equation:

$$X''(x) + \lambda X(x) = 0$$

Integrate:

$$\int_0^L X(x) X''(x)\, dx + \lambda \int_0^L X(x)^2\, dx = 0$$

$$[X'(x)X(x)]_0^L - \int_0^L \left(X'(x)\right)^2 dx + \lambda \int_0^L X(x)^2\, dx = 0$$

From the boundary conditions ($X'(0) = X'(L) = X(0) = X(L) = 0$) we have:

$$[X'(x)X(x)]_0^L - \int_0^L \left(X'(x)\right)^2 dx + \lambda \int_0^L X(x)^2\, dx = 0$$

$$-\int_0^L \left(X'(x)\right)^2 dx + \lambda \int_0^L X(x)^2\, dx = 0$$

$$\lambda = \frac{\int_0^L \left(X'(x)\right)^2 dx}{\int_0^L X(x)^2\, dx}$$

Which being the ratio of two positive numbers, is always positive. Hence $\lambda > 0$.

Proof that Distinct Eigenfunctions are Orthogonal

Begin with two distinct eigenvalues of the differential operator:

$$X_\lambda''(x) + \lambda X_\lambda(x) = 0$$
$$X_\mu''(x) + \mu X_\mu(x) = 0$$

Multiply each equation by the other eigenfunction:

$$X_\lambda''(x)X_\mu(x) + \lambda X_\lambda(x)X_\mu(x) = 0$$
$$X_\mu''(x)X_\lambda(x) + \mu X_\mu(x)X_\lambda(x) = 0$$

Integrate both expressions over $(0, L)$:

$$\int_0^L X_\lambda''(x)X_\mu(x) + \lambda X_\lambda(x)X_\mu(x)\, dx = \left[X_\mu(x)X_\lambda'(x)\right]_0^L - \int_0^L X_\lambda'(x)X_\mu'(x) + \lambda X_\lambda(x)X_\mu(x)\, dx$$
$$\int_0^L X_\mu''(x)X_\lambda(x) + \mu X_\mu(x)X_\lambda(x)\, dx = \left[X_\lambda(x)X_\mu'(x)\right]_0^L - \int_0^L X_\lambda'(x)X_\mu'(x) + \mu X_\lambda(x)X_\mu(x)\, dx$$

Subtract the resulting expressions:

$$\left[X_\mu(x)X_\lambda'(x)\right]_0^L - \int_0^L X_\lambda'(x)X_\mu'(x) + \lambda X_\lambda(x)X_\mu(x)\, dx - \left[X_\lambda(x)X_\mu'(x)\right]_0^L$$
$$+ \int_0^L X_\lambda'(x)X_\mu'(x) + \mu X_\lambda(x)X_\mu(x)\, dx = 0$$
$$= \left[X_\mu(x)X_\lambda'(x)\right]_0^L - \left[X_\lambda(x)X_\mu'(x)\right]_0^L - \int_0^L \lambda X_\lambda(x)X_\mu(x)\, dx + \int_0^L \mu X_\lambda(x)X_\mu(x)\, dx$$
$$= \left[X_\mu(x)X_\lambda'(x)\right]_0^L - \left[X_\lambda(x)X_\mu'(x)\right]_0^L + (\mu - \lambda)\int_0^L X_\lambda(x)X_\mu(x)\, dx$$

For Dirichlet, Neumann, or periodic boundary conditions, the first two terms will be zero, hence:

$$(\mu - \lambda)\int_0^L X_\lambda(x)X_\mu(x)\, dx = 0$$

If the eigenvalues are distinct, $\mu \neq \lambda$, and hence the orthogonality condition follows:

$$\int_0^L X_\lambda(x) X_\mu(x)\,dx = 0$$