Classical Physics

Lagrangian

Generalised momentum

$$p_i(t) = \frac{\partial T(t)}{\partial \dot{q}_\iota}$$

Differentiating the generalised momentum with respect to time yields a set of equations (one for each generalised coordinate). We call these the Lagrange equations.

$$L = K - V$$

Lagrange equation for conservative forces

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\iota}\right) = \frac{\partial L}{\partial q_i}$$

We need to get a system of equations that defines the behaviour of the system over time. Three ways to solve for equations of motion:

- Substitute L into Lagrange equations and solve directly
- Substitute L into Lagrange equations and find constants of the motion
- Derive equations directly by taking $\frac{dp_i}{dt}$

Finding Constants of the Motion

If we find an ignorable coordinate q_k, then a constant of the motion is:

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

If the Lagrangian is explicitly independent of time, then the Hamiltonian will be a constant of the motion

$$H = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

Rotational Motion

$$K = \frac{1}{2} m v^2 + \frac{1}{2} \Omega' I \Omega$$

Velocity of rotating object (r is distance from origin of rotation to CM). Note that the origin of rotation and the CM will in general be different points.

$$v = r\omega = r\dot{\phi}$$

This same formula can also be used to find the magnitude of Ω

$$v = r\Omega$$
$$\Omega = \frac{v}{r}$$

For a cone rolling on a table, the angular velocity vector Ω points along the line of contact between cone and table.

Individual elements of the Ω vector can be found by taking the projection of Ω onto the different dimensions of the principal axes.

Moment of inertia:

$$I_{xx} = \iiint \rho(y^2 + z^2)\, dV$$

Centre of mass along z axis:

$$CM_z = \frac{1}{M} \iiint \rho z\, dV$$

Angular Momentum

$$M = I\Omega$$

$$\frac{de_i}{dt} = \Omega \times e_i$$
$$e_1 \times e_2 = e_3$$
$$\therefore e_2 \times e_1 = -e_3$$
$$e_3 \times e_1 = e_2$$

Coplanar Vectors

Three vectors Ω, M, e_3 are coplanar if $(\Omega \times M) \cdot e_3 = 0$

Symmetry

Symmetry occurs when the centre of mass lies along one of the principal axes of the body. If symmetry exists it is usually very easy to find the principal axes. If not, one must solve the Eigenvalue problem:

$$S^{-1}IS = \Lambda$$

Choosing S (the matrix of eigenvectors) such that Λ (the matrix of eigenvalues) is diagonal.

Precession

Precession occurs when the angular velocity vector does not lie along one of the principal axes of the body.

The precession period of a rotating biaxial body is given by:

$$T = \frac{2\pi}{\epsilon\Omega_3}$$

Oscillators

For a single spring

First write the Lagrangian in terms of x, solve for Lagrange equations, and evaluate these equations to solve for $x = x_e$ at equilibrium (i.e. when $\ddot{x} = 0$). Then define $\xi = x - x_e$, and substitute this into the Lagrange equation. It should now be possible to find an oscillatory solution for the lagrange equation.

For two springs

Find the Lagrange equations in terms of x, and then rewrite them with the following substitution:

$$x_1 = \xi_1 + b$$
$$x_2 = \xi_2 - b$$

For three springs

Find the Lagrange equations in terms of x, and then rewrite them with the following substitution:

$$x_1 = \xi_1 + b$$
$$x_2 = b$$
$$x_3 = \xi_3 - b$$

Oscillatory solutions for a differential equation take the form:

$$\xi = Ae^{-i\omega t}$$

Where ξ is defined as the displacement from equilibrium. Coupled oscillators must have the same ω as otherwise the motion would cancel itself out.

To solve for ω, find the equations of motion in terms of ξ_i and $\dot{\xi}_i$, write these in matrix form, and then equate the determinant of the square matrix to zero. This guarantees the matrix is not reducible, and so will have solutions we desire.

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