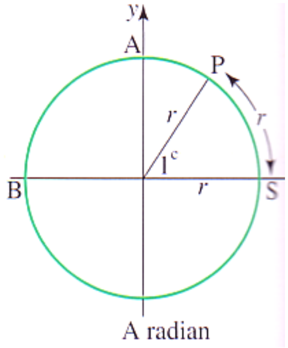
Trigonometry

Angles

The constant $\pi = 3.14159$ represents the ratio between the diameter and circumference of any circle. Thus, a circle with a radius of 1 unit has a circumference of 2π.



Therefore it is evident that 1 radian is equal to $\frac{1}{2\pi}$, or slightly under 1/6th of the circumference of a unit circle.

When we measure angles in relation to a circle, we are actually measuring the circumference of that circle. Thus, an angle of $1°$ simply represents $\frac{1}{360}$th of the circumference of the circle.

Putting these two measurements together, we see that $360° = 2\pi^c$. Therefore it is evident that:

$$1° = \frac{2\pi}{360}^{c}, \qquad 1^c = \frac{360°}{2\pi}$$

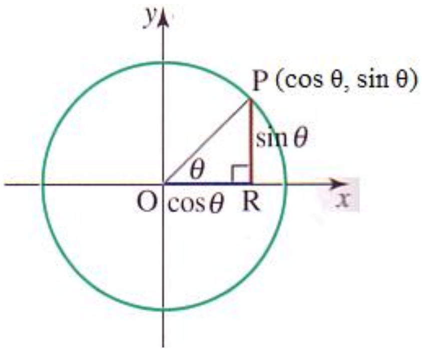
Trigonometric Operators

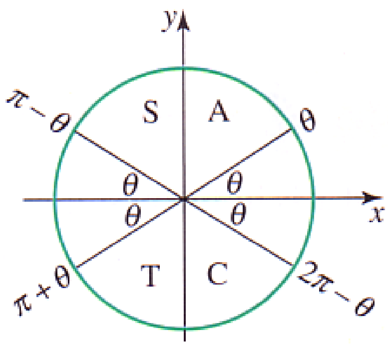
As we move around the circumference of a unit circle a distance of θ degrees or radians, we can express our position in terms of (x, y) corordinates. This can be done using the special operators sine and cosine.

Note that because the circle is symmetrical in all four quadrants, different values of θ will produce the same (x, y) coordinates, although owing to shifts above and below the axes, the sign may differ.

Sine

$$\sin\theta = y = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\csc\theta}$$





Note that in Q3 we use the formula $\theta - \pi$

Cosine

$$\cos\theta = x = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sec\theta}$$

Tangent

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} = \frac{\text{opposite}}{\text{adjacent}}$$

Cotangent

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{x}{y} = \frac{\text{adjacent}}{\text{opposite}}$$

Example 1

Given that $\cos x = 0.8$, find $\cos(3\pi - x)$:

$$\begin{aligned}\cos(3\pi - x) &= \cos(2\pi + \pi - x)\\ &= \cos(\pi - x)\\ &= -\cos(x)\\ &= -0.8\end{aligned}$$

Note that we must first subtract off the additional multiples of 2π (which represent additional full rotations around the circle), and then convert the remaining angle into a first quadrant angle.

Example 2

Find the exact value of the following angle:

Step 1: Convert to positive angle

$$\tan\left(-\frac{5\pi}{6}\right) = \tan\left(-\frac{5\pi}{6} + 2\pi\right)$$
$$= \tan\left(-\frac{5\pi}{6} + \frac{12\pi}{6}\right)$$
$$= \tan\left(\frac{7\pi}{6}\right)$$

Step 2: Rewrite in solvable form

$$\tan\left(\frac{7\pi}{6}\right) = \frac{\sin\left(\frac{7\pi}{6}\right)}{\cos\left(\frac{7\pi}{6}\right)}$$

Step 3: Convert to first quadrant (basic) angle

$$= \frac{-\sin\left(\frac{7\pi}{6} - \pi\right)}{-\cos\left(\frac{7\pi}{6} - \pi\right)}$$
$$= \frac{-\sin\frac{\pi}{6}}{-\cos\frac{\pi}{6}}$$

Step 4: Solve and simplify

$$= \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$$
$$= -\frac{1}{2} \times -\frac{2}{\sqrt{3}}$$
$$= \frac{1}{\sqrt{3}}$$

First we had to convert the negative angle to a positive angle by adding 2π, then we expressed the tangent angle as a relationship between sin and cos angles, and then we converted these angles to first quadrant angles with known exact values, and then simplify.

Example 3

Solve the following expression using the half angle identities:

Step 1: Convert to positive angle

$$\cos\left(-\frac{3\pi}{8}\right) = \cos\left(\frac{-3\pi}{8} + \frac{16\pi}{8}\right)$$
$$= \cos\left(\frac{13\pi}{8}\right)$$

Step 2: Rewrite in solvable form

$$\cos\left(\frac{13\pi}{8}\right) = \sqrt{\cos^2\left(\frac{1}{2}\times\frac{13\pi}{4}\right)}$$

$$= \sqrt{\frac{\left(1+\cos\left(\frac{13\pi}{4}\right)\right)}{2}}$$

Step 3: Convert to first quadrant (basic) angle

$$= \sqrt{\frac{\left(1+\cos\left(\frac{13\pi}{4}-\frac{8\pi}{4}\right)\right)}{2}}$$

$$= \sqrt{\frac{\left(1+\cos\left(\frac{5\pi}{4}\right)\right)}{2}}$$

$$= \sqrt{\frac{\left(1-\cos\left(\frac{5\pi}{4}-\frac{4\pi}{4\pi}\right)\right)}{2}}$$

$$= \sqrt{\frac{\left(1-\cos\left(\frac{\pi}{4}\right)\right)}{2}}$$

Step 4: Solve and simplify

$$= \sqrt{\frac{\left(1-\frac{1}{\sqrt{2}}\right)}{2}}$$

$$= \sqrt{\left(\frac{1}{2}-\frac{1}{2\sqrt{2}}\right)}$$

$$= \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$

Note that by squaring the angle, we were able to convert it into a problem solvable using the 'Half Angle Identity' method.

Example 4

Find all the solutions to the equation $\sqrt{2}\cos x + 1 = 0$ in the domain $0 \le x \le 4\pi$

Step 2: Rewrite in solvable form

$$\sqrt{2}\cos x + 1 = 0$$

$$\cos x = -\frac{1}{\sqrt{2}}$$

Step 3: Find basic angle

$$\therefore \text{basic angle} = \frac{\pi}{4}$$

Step 4: Convert to correct quadrants

$$\cos x < 0, \therefore Q2\ \&\ Q3$$

$$x = \left(\pi - \frac{\pi}{4}\right), \left(\pi + \frac{\pi}{4}\right)$$
$$= \left(\frac{4\pi}{4} - \frac{\pi}{4}\right), \left(\frac{4\pi}{4} + \frac{\pi}{4}\right)$$
$$= \left(\frac{3\pi}{4}\right), \left(\frac{5\pi}{4}\right)$$

Step 5: Expand to fill domain

$$x = \left(\frac{3\pi}{4}\right), \left(\frac{5\pi}{4}\right), \left(\frac{3\pi}{4} + \frac{8\pi}{4}\right), \left(\frac{5\pi}{4} + \frac{8\pi}{4}\right)$$
$$= \left(\frac{3\pi}{4}\right), \left(\frac{5\pi}{4}\right), \left(\frac{11\pi}{4}\right), \left(\frac{13\pi}{4}\right)$$

Note that the answer is expressed in the same form as the question, in this case in radians

Example 5

Find all the solutions to the equation $\sin 3x = \cos 3x$ in the domain $0 \leq x \leq 2\pi$

Step 1: Alter domain

$$0 \leq x \leq 2\pi$$
$$0 \leq 3x \leq 6\pi$$

Step 2: Rewrite in solvable form

$$\sin 3x = \cos 3x$$
$$\frac{\sin 3x}{\cos 3x} = \frac{\cos 3x}{\cos 3x}$$
$$\tan 3x = 1$$

Step 3: Find basic angle

$$let\ \theta = 3x$$
$$\sec^2\theta - \tan^2\theta = 1$$
$$\sec^2\theta - 1 = 1$$
$$\sec^2\theta = 2$$
$$\sec\theta = \sqrt{2}$$
$$\cos\theta = \frac{1}{\sqrt{2}}$$
$$\theta = \frac{\pi}{4}$$
$$\therefore \text{basic angle} = \frac{\pi}{4}$$

Step 4: Convert to correct quadrants

$$\tan 3x > 0, \therefore Q1\ \&\ Q3$$
$$3x = \left(\frac{\pi}{4}\right), \left(\pi + \frac{\pi}{4}\right)$$
$$3x = \frac{\pi}{4}, \frac{5\pi}{4}$$

Step 5: Expand to fill domain

$$3x = \left(\frac{\pi}{4}\right), \left(\frac{5\pi}{4}\right), \left(\frac{\pi}{4} + \frac{8\pi}{4}\right), \left(\frac{5\pi}{4} + \frac{8\pi}{4}\right) \left(\frac{\pi}{4} + \frac{16\pi}{4}\right), \left(\frac{5\pi}{4} + \frac{16\pi}{4}\right)$$

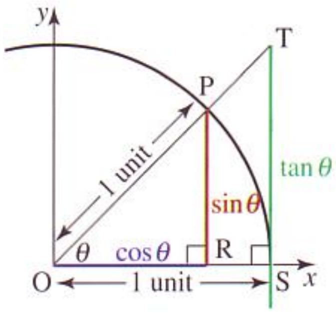
$$3x = \left(\frac{\pi}{4}\right), \left(\frac{5\pi}{4}\right), \left(\frac{9\pi}{4}\right), \left(\frac{13\pi}{4}\right), \left(\frac{17\pi}{4}\right), \left(\frac{21\pi}{4}\right)$$

Step 6: Solve for unknown

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12}$$

Pythagorean Identities

Pythagoras' theorem tells us that for a right-angled triangle: $Hyp^2 = Adj^2 + Opp^2$. As $\cos\theta = Adj$ and $\sin\theta = Opp$, and as we know that Hyp must always equal 1, we can derive the following identities:



Identity 1

$\cos^2\theta + \sin^2\theta = 1$

Identity 2

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$\sec^2\theta - \tan^2\theta = 1$$

Identity 3

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$
$$\cot^2\theta + 1 = \csc^2\theta$$
$$\csc^2\theta - \cot^2\theta = 1$$

Example 6

Simplify the following expression using the Pythagorean identities:

$$(\cos x + \sin x)^2 + (\cos x - \sin x)^2$$
$$= (\cos^2 x + 2\cos x \sin x + \sin^2 x) + (\cos^2 x - 2\cos x \sin x + \sin^2 x)$$
$$= (1 + 2\cos x \sin x) + (1 - 2\cos x \sin x)$$
$$= 2 + 2\cos x \sin x - 2\cos x \sin x$$
$$= 2$$

Example 7

Simplify the following expression using the Pythagorean identities:

$$\frac{1+\cot x}{\csc x} - \frac{1+\tan x}{\sec x} = \frac{\sec x\,(1+\cot x)}{\csc x \sec x} - \frac{\csc x\,(1+\tan x)}{\csc x \sec x}$$
$$= \frac{\sec x\,(1+\cot x) - \csc x\,(1+\tan x)}{\csc x \sec x}$$
$$= \frac{\left(\frac{1}{\cos x}x + \frac{\cot x}{\cos x}\right) - \left(\frac{1}{\sin x} + \frac{\tan x}{\sin x}\right)}{\frac{1}{\sin x} \times \frac{1}{\cos x}}$$
$$= \sin x \cos x \left(\left(\frac{1+\cot x}{\cos x}\right) - \left(\frac{1+\tan x}{\sin x}\right)\right)$$
$$= \sin x \cos x \left(\frac{\sin x\,(1+\cot x)}{\sin x \cos x} - \frac{\cos x\,(1+\tan x)}{\sin x \cos x}\right)$$

$$= \sin x \cos x \left(\frac{\sin x\,(1 + \cot x) - \cos x\,(1 + \tan x)}{\sin x \cos x} \right)$$
$$= \sin x \left(1 + \frac{\cos x}{\sin x}\right) - \cos x \left(1 + \frac{\sin x}{\cos x}\right)$$
$$= (\sin x + \cos x) - (\cos x + \sin x)$$
$$= \sin x + \cos x - \cos x - \sin x$$
$$= 0$$

Note that by simplifying out the fractions and expressing all trigonometric values in terms of sin and cos, it was possible to simplify the above expression.

Other Useful Formulae

Complementary Angles

Complementary angles add to $90°$ or $\frac{\pi}{2}$ radians, for example $\frac{\pi}{3}$ and $\frac{\pi}{6}$ are complementary

If angles α and θ are complementary, then $\sin\alpha = \cos\theta$ and $\cos\alpha = \sin\theta$

If angles α and θ are complementary, then $\tan\alpha = \cot\theta$ and $\cot\alpha = \tan\theta$

Sum and Difference Laws

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Double Angle Identities

$\sin 2x = 2 \sin x \cos x$
$\cos 2x = \cos^2 x - \sin^2 x$

Half Angle Identities

$\sin^2\left(\frac{y}{2}\right) = \dfrac{1 - \cos y}{2}$
$\cos^2\left(\frac{y}{2}\right) = \dfrac{1 + \cos y}{2}$

Exact Values

Angle	$0°$ 0	$30°$ $\frac{\pi}{6}$	$45°$ $\frac{\pi}{4}$	$60°$ $\frac{\pi}{3}$	$90°$ $\frac{\pi}{2}$
$\sin\theta$	0	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	UD

Example 8

If $\sin\theta = \frac{12}{13}$ and $\frac{\pi}{2} < \theta < \pi$, find $\cos\theta$ and hence find $\tan\theta$

$$\sin\theta = \frac{12}{13} = \frac{Opp}{Hyp}$$

$$\therefore Hyp^2 = Opp^2 + Adj^2$$

$$13^2 = 12^2 + c^2$$

$$169 = 144 + c^2$$

$$25 = c^2$$

$$c = 5$$

$$\frac{\pi}{2} < \theta < \pi \therefore \cos\theta < 0$$

$$\therefore \cos\theta = -\frac{Adj}{Hyp}$$

$$= -\frac{5}{13}$$

$$\frac{\pi}{2} < \theta < \pi \therefore \tan\theta < 0$$

$$\therefore \tan\theta = -\frac{Opp}{Adj}$$

$$= -\frac{12}{5}$$

Example 9

Simplify the following expression:

$$\begin{aligned}\sin 6x \cos 2x - \cos 6x \sin 2x &= \sin(6x - 2x) \\ &= \sin 4x \\ &= 2\sin 2x \cos 2x\end{aligned}$$

Note that here we used a difference law and a double angle identity.

Example 10

Simplify the following expression:

$$\begin{aligned}\frac{\sin 2x}{1 - \sin^2 x} &= \frac{2\sin x \cos x}{\cos^2 x} \\ &= \frac{2\sin x \cos x}{\cos x \cos x} \\ &= \frac{2\sin x}{\cos x} \\ &= 2\tan x\end{aligned}$$

Note that here we used a Pythagorean identity and a double angle identity.

Example 11

Simplify the following expression:

$$\begin{aligned}2\sin 3x \cos 3x \cos 5x - (\cos^2 3x - \sin^2 3x)\sin 5x &= 2\sin 3x \cos 3x \cos 5x - \cos 6x \sin 5x \\ &= \sin 6x \cos 5x - \cos 6x \sin 5x \\ &= \sin(6x - 5x) \\ &= \sin x\end{aligned}$$

Note that here we used sum and difference laws and double angle identities.