Functions and Graphs

Sets, Functions and Relations

Set Notation

A set is a collection of objects. The objects that are in the set are known as the elements or members of the set. Membership of a set is expressed thus: $A = \{1, 2, 3, 4\}$

$x \in A$: x is a member of set A

$x \notin A$: x is not a member of set A

$B \subseteq A$: B is a subet of A, meaning that all members of B are also members of A

$A \cup B$: the union of sets *A* and *B*, the set of elements that are either in *A* or in *B*

$A \cap B$: the intersection of A and B, the set of elements that are members of both A and B

The set $\emptyset$ is called the empty set or null set

$A \cap B = \emptyset$: sets *A* and *B* have no elements in common; *A* and *B* are disjoint

In mathematical logic, a colon is used as an alternative to a vertical bar, to mean "such that"

$S = \{x \in R: 1 < x < 3\}$ means "*S* is the set of all *x* in **R** (the real numbers) such that *x* is strictly greater than 1 and strictly smaller than 3"

Types of Numbers

The set of all natural numbers is given by **N**.

$$N = \{1, 2, 3, 4 \dots\}$$

The set of all integers is given by **Z**.

$$Z = \{\dots - 3, -2, -1, 0, 1, 2, 3 \dots\}$$

The set of all rational numbers is given by **Q**, and is comprised of all numbers are those which can be represented by either a finite number, or recurring pattern, of decimal places. In other words, these are real numbers that can be written as a fraction.

$$Q = \{\tfrac{p}{q} : p \in Z \text{ and } q \in N\}$$

The set of real numbers is represented by **R**, and includes all numbers that are not imaginary

$$R = \{1, 67, \sqrt{3}, -4, {}^{3}/_{4}, \pi \dots\}$$

Note that $N \subseteq Z \subseteq Q \subseteq R$

Domain and Range

An ordered pair, denoted (a, b), is a pair of elements *a* and *b* in which *a* is considered to be the first element and *b* the second. A relation is a set of ordered pairs.

The domain of a relation *S* is the set of all first elements of the ordered pairs in *S*.
The range of a relation *S* is the set of all second elements of the ordered pairs in *S*.
For example, the set $\{(x,y): y = x+1, x \in \{1,2,3,4\}\}$ is the relation $\{(1,2),(2,3),(3,4),(4,5)\}$

The *domain* of a function is the set of all possible *input* values (generally x) for that function
The *range* of a function is the set of all possible *output* values (generally y) for that function

$$\text{If } f(x) \text{ is defined when } -3 < x < 3, \text{then Domain} = (-3,3)$$
$$\text{If } f(x) \text{ is defined when } -3 \leq x \leq 3, \text{then Domain} = [-3,3]$$
$$\text{If } f(x) \text{ is defined when } x > 0, \text{then Domain} = R^+ = (0,\infty)$$
$$\text{If } f(x) \text{ is defined when } x \neq 0, \text{then Domain} = R\backslash\{0\}$$

Implied Domain

When the domain of a relation is not explicitly stated, it is understood to consist of all real numbers for which the defining rule has meaning. For example:

$S = \{(x,y): y = x^2\}$ is assumed to have domain *R* and

$T = \{(x,y): y = \sqrt{x}\}$ is assumed to have domain [0,∞).

Relations and Functions

A function is a relation such that no two ordered pairs of the relation have the same first element. This can be determined by graphing the relation and then determining if a vertical line can only intersect the relation once; if so then the relation is a function.

Functions are usually denoted by lower case letters such as *f*, *g*, *h*.

This definition of a function tells us that for each x in the domain of f there is a unique element y in the range such that $(x,y) \in f$.

For the function $\{(x,y): y = x^2\}$, write:

$$f: R \to R, f(x) = x^2$$

Here the first R refers to the domain, and the second R refers to the codomain, which is simply the restriction of the values one permits the output of the function (y) to take. Usually it is simply R.

This would be read "f is a function such that it has a domain of R and a codomain of R, and f of x is equal to the square of x"

Adding Functions and Composite Functions

If we have two functions f(x) and f(y), the domain of both *f* + *g* and *fg* is the intersection of the domains of *f* and *g*, i.e. the values of *x* for which both *f* and *g* are defined

For the composite function $g(f(x))$ to be defined, the range of *f* must be a subset ⊆ of the domain of *g*.

One-to-One Functions

A function is said to be one-to-one if every value of a has its own unique corresponding value of b. The test for this is drawing a horizontal line across the function and seeing if it intersects only once. If so it is a one-to-one function.

Inverses

The inverse of a function is denoted $f^{-1}(x)$, and has the property such that:

$$f^{-1}(f(x)) = f(f^{-1}(x)) = x$$

Given that a relation is a set of ordered pairs, the inverse of a relation involves simply swapping those pairs around. For example the inverse relation of $\{(1,2), (1,3), (4,2), (4,3), (1,5)\}$ is simply {(2, 1), (3, 1), (2, 4), (3, 4), (5, 1)}.

The inverse of a relation or function can be found by interchanging the x and y values, and simplifying to make y the subject. When changing a function to its inverse, the range and domain and swapped.

A graph can be converted to its inverse by reflecting it through the line $y = x$

A graph centred at $(0,0)$ is shifted to being centred on (a, b) when:

$$y = f(x) \text{ becomes } (y - b) = f(x - a)$$

$$\text{dom } f^{-1} = \text{ran } f$$
$$\text{ran } f^{-1} = \text{dom } f$$

A function which is one-to-one has an inverse function. A function which is many-to-one has an inverse relation that is not a function.

Absolute Values

Absolute value functions can be sketched by reflecting all points where $f(x) < 0$ in the x-axis

Absolute value functions can be transformed just like other graphs, even when such transformations reintroduce negative values.

$$y = a|f(x)| - k$$

This function is dilated by a factor of a in the y direction and translated k units down

Absolute value functions usually have a range of R^+, and often display sharp breaks on the x-axis called cusps

Linear Graphs

Linear graphs are polynomials of degree 1.

- General equation: $y = mx + c$
- Intercept form: $bx + ay = ab$
- Gradient: $m = \frac{y_2 - y_1}{x_2 - x_1}$
- Equation of line passing through (x_1, y_1): $y - y_1 = m(x - x_1)$
- If a graph is perpendicular to a graph with gradient m, then its gradient will be $-\frac{1}{m}$

Example 1

Find the equation of the line with gradient 2 which passes through $(3, -2)$

$$y - y_1 = m(x - x_1)$$
$$y - (-2) = 2(x - 3)$$

$$y + 2 = 2x - 6$$
$$y = 2x - 8$$

Example 2

Find the equation of the line passing through the points $(0,8)$ and $(-2,2)$

$$y - 8 = \left(\frac{2-8}{-2-0}\right)(x - 0)$$
$$y - 8 = \left(\frac{-6}{-2}\right)(x)$$
$$y - 8 = 3x$$
$$y = 3x + 8$$

Example 3

What is the range and domain of $y = 1 - 2x, x \in (-\infty, -1)$?

Domain $=$ set of $x = (-\infty, -1)$

$$y = 1 - 2(-1)$$
$$= 1 + 2$$
$$= 3$$
$$y = 1 - 2(-1000)$$
$$= 1 + 2000$$
$$= 2001$$

$\therefore$ Range $=$ set of $y = (3, \infty)$

Quadratic Graphs

Quadratic graphs are polynomials of degree 2.

General equation: $y = ax^2 + bx + c, x \in R$

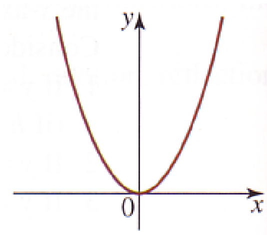
Turning Point Form

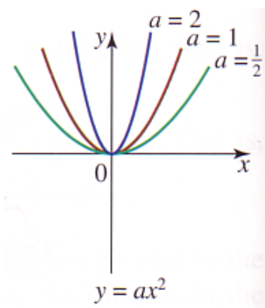
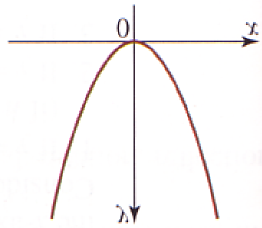
The basic formula is $y = a(x - h)^2 + k$

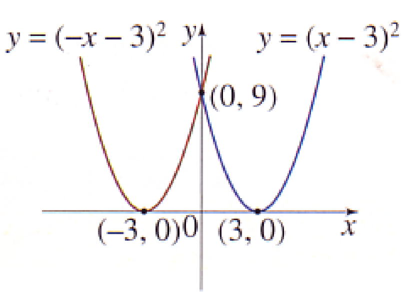
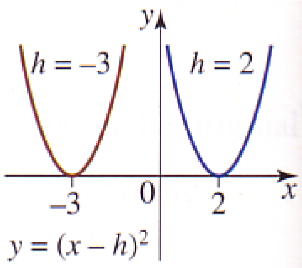
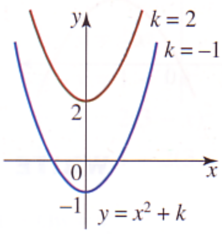
Range: R

Domain: R

Turning point occurs at (h, k)



Transformation	Formula	Graph
Dilation in y-direction	$y = ax^2$	
Reflection in the x-axis	$y = -ax^2$	

Reflection in the y-axis	$y = (-x)^2$	
Horizontal translation	$y = (x - h)^2$	
Vertical translation	$y = x^3 + k$	

Factor Form

The basic formula is $y = ax^2 + bx + c$

- For $a > 0$, the graph has a minimum value, for $a < 0$, the graph has a maximum value
- The y-intercept is given by c
- The x-intercepts can be found by solving the equation $ax^2 + bx + c = 0$
- This can be done either by using the quadratic formula
- The basic form can be converted to turning point form by completing the square (take half of x coefficient and sign, and square it, adding the result both inside and outside the brackets)

Example 1

Calculate all intercepts of the function $f(x) = 12 - 5x - 2x^2$

Y-intercept occurs when $x = 0$

$f(0) = 12 - 5(0) - 2(0)^2$

$= 12$

$\therefore$ Y-intercept $= (0,12)$

X-intercept occurs when $y = 0$

Use quadratic formula

$12 - 5x - 2x^2 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(-2)(12)}}{2(-2)}$$

$x = \frac{5 \pm \sqrt{25+96}}{-4}$
$x = \frac{5 \pm \sqrt{121}}{-4}$
$x = \frac{5 \pm 11}{-4}$
$x = \frac{5+11}{-4}, x = \frac{5-11}{-4}$
$x = -4, x = -1.5$
∴ X-intercepts $= (-1.5,0), (-4,0)$

Example 2

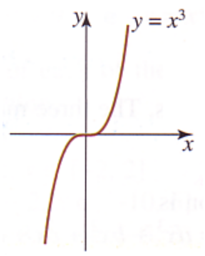
Find the turning point of the function $y = 3 + 8x - 2x^2$
$y = -2x^2 + 8x + 3$
Complete the square
$y = -2\left(x^2 - 4x - \frac{3}{2}\right)$
$y = -2\left(\left[x^2 - 4x + \left(\frac{-4}{2}\right)^2\right] - \frac{3}{2} - \left(\frac{-4}{2}\right)^2\right)$
$y = -2\left([x^2 - 4x + 4] - \frac{3}{2} - 4\right)$
$y = -2\left((x-2)^2 - \frac{3}{2} - \frac{8}{2}\right)$
$y = -2\left((x-2)^2 - \frac{11}{2}\right)$
$y = -2(x-2)^2 + 11$
∴ Turning point $= (2,11)$

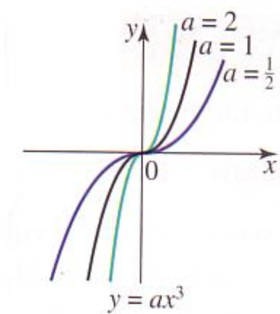
Cubic Graphs

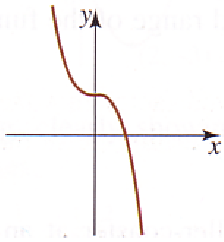
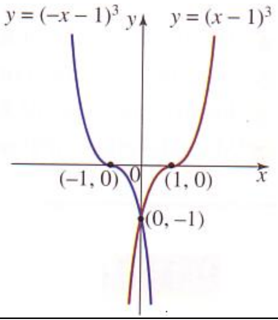
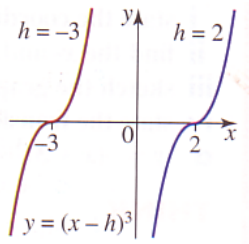
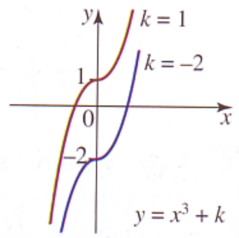
Cubic graphs are polynomials of degree 3.
General equation: $y = ax^3 + bx^2 + cx + d, x \in R$

Basic Form

The basic formula is $y = a(x-h)^3 + k$
Note that it can also take the form $y = ax^3 + bx$
Range: R
Domain: R
Stationary point of inflection occurs at $(-h, k)$



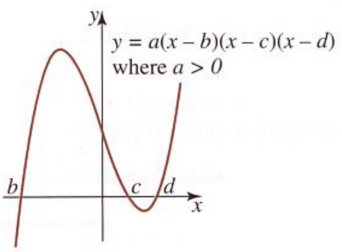
Transformation	Formula	Graph
Dilation in y-direction	$y = ax^3$	

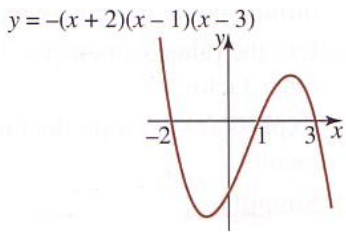
Reflection in the x-axis	$y = -ax^3$	
Reflection in the y-axis	$y = (-x)^3$	
Horizontal translation	$y = (x - h)^3$	
Vertical translation	$y = x^3 + k$	

Factor Form

The basic formula is $y = a(x - b)(x - c)(x - d)$

- The graph will have x-intercepts at $b, c,$ and d
- It has two turning points, one above and one below the x-axis
- It is dilated in the y-direction by a factor of a
- If $a < 0$, the graph will be reflected in the x-axis





Example

Find all the intercepts of $y = x^3 - x^2 - 10x - 8$

X-intercept occurs when $y = 0$

Use factor theorem

$(1)^3 - (1)^2 - 10(1) - 8 = 1 - 1 - 10 - 8$
$= -18$
$\neq 0, \therefore (x-1)$ is not a factor

$(-1)^3 - (-1)^2 - 10(-1) - 8 = -1 - 1 + 10 - 8$
$= 0, \therefore (x+1)$ is a factor

Use long division

$$\begin{array}{r l}
 & \underline{x^2 - 2x - 8} \\
(x+1) & |x^3 - 1x^2 - 10x - 8 \\
 & \underline{-(x^3 + 1x^2)} \\
 & -2x^2 - 10x - 8 \\
 & \underline{-(-2x^2 - 2x)} \\
 & -8x - 8 \\
 & \underline{-(-8x - 8)} \\
 & 0
\end{array}$$

$\therefore x^2 - 2x - 8$ is a factor

Use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-8)}}{2(1)}$$
$$x = \frac{2 \pm \sqrt{4 - 4(-8)}}{2}$$
$$x = \frac{2 \pm \sqrt{4 + 32}}{2}$$
$$x = \frac{2 \pm \sqrt{36}}{2}$$
$$x = \frac{2 \pm 6}{2}$$
$$x = \frac{2 - 6}{2}, x = \frac{2 + 6}{2}$$
$x = -2, x = 4$

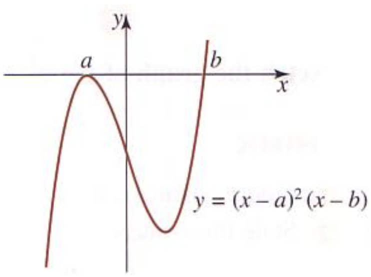
$\therefore (x+2)$ and $(x-4)$ are factors

$y = (x+1)(x+2)(x-4)$

Repeated Factor Form

The basic formula is $y = a(x-b)^2(x-c)$

- The graph will have x-intercepts at b and c
- It has two turning points, one on the x-axis at b
- It is dilated in the y-direction by a factor of a
- If $a < 0$, the graph will be reflected in the x-axis

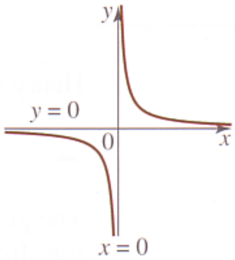


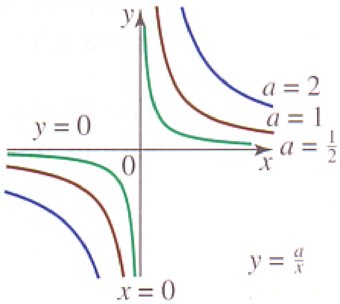
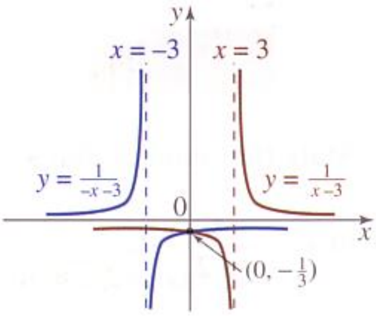
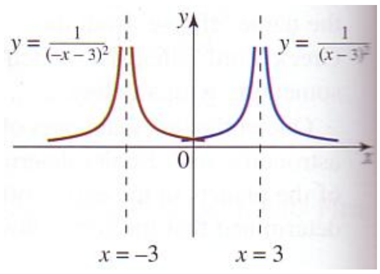
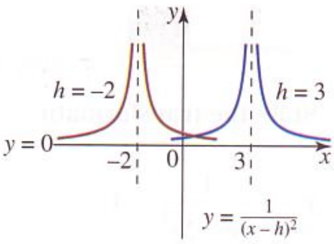
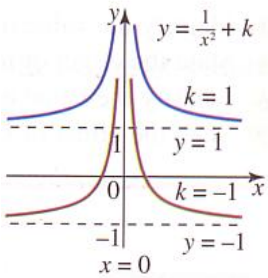
Other Graphs

The Hyperbola

The basic formula is $y = \frac{a}{(x-h)} + k$

Range: $R\backslash\{0\}$
Domain: $R\backslash\{0\}$
Asymptotes occur at $x = h$ and $y = k$

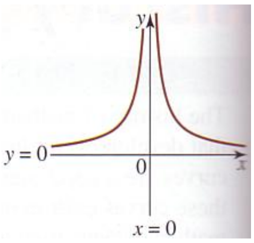


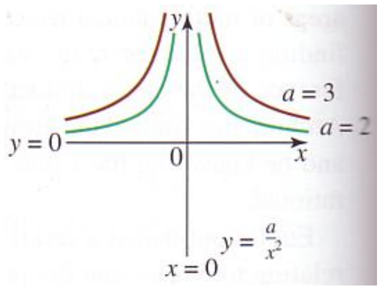
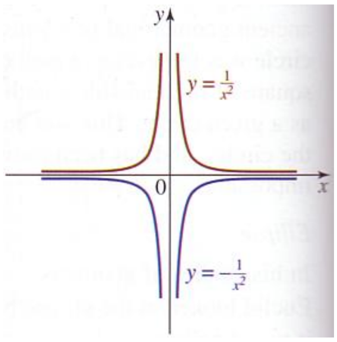
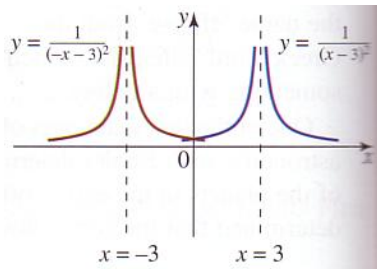
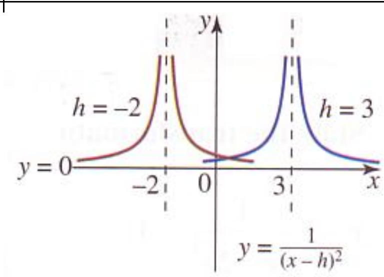
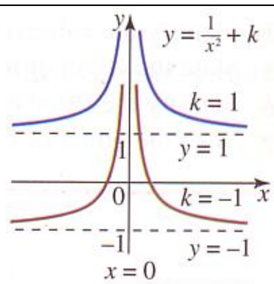
Transformation	Formula	Graph
Dilation in y-direction	$y = \frac{a}{x}$	
Reflection in the x-axis	$y = \frac{-a}{x}$	
Reflection in the y-axis	$y = \frac{a}{(-x)}$	
Horizontal translation	$y = \frac{a}{(x-h)}$	
Vertical translation	$y = \frac{a}{x} + k$	

The Truncus

The basic formula is $y = \frac{a}{(x-h)^2} + k$

Range: $[0, \infty)$
Domain: $R\backslash\{0\}$
Asymptotes occur at $x = h$ and $y = k$

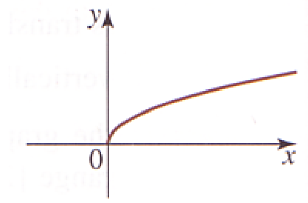


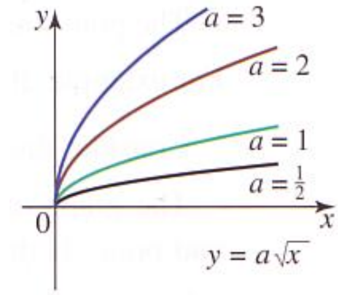
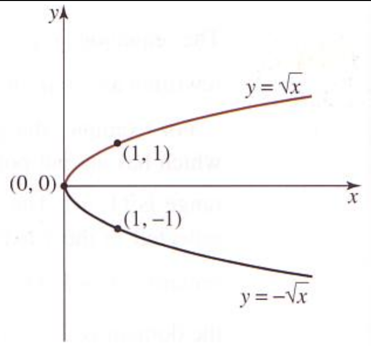
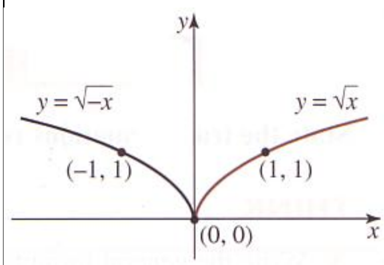
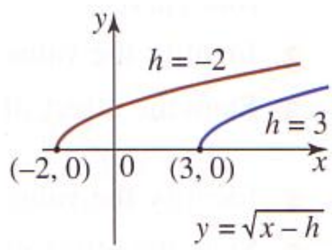
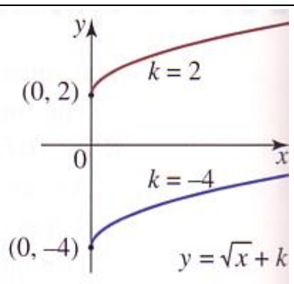
Transformation	Formula	Graph
Dilation in y-direction	$y = \frac{a}{x^2}$	
Reflection in the x-axis	$y = \frac{-a}{x^2}$	
Reflection in the y-axis	$y = \frac{a}{(-x)^2}$	
Horizontal translation	$y = \frac{a}{(x-h)^2}$	
Vertical translation	$y = \frac{a}{x^2} + k$	

Square Root Function

The basic formula is $y = a\sqrt{x-h} + k$

Range: $[0, \infty)$
Domain: $[0, \infty)$



Transformation	Formula	Graph
Dilation in y-direction	$y = a\sqrt{x}$	
Reflection in the x-axis	$y = -\sqrt{x}$	
Reflection in the y-axis	$y = \sqrt{-x}$	
Horizontal translation	$y = \sqrt{x-h}$	
Vertical translation	$y = \sqrt{x} + k$	

Trigonometric Graphs

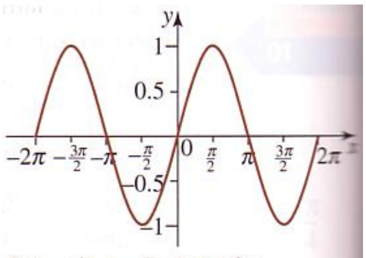
Sine and Cos Functions

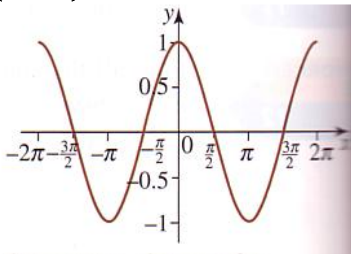
The basic formula is $y = a\sin n(x-b)+c, y = a\cos n(x-b)+c$

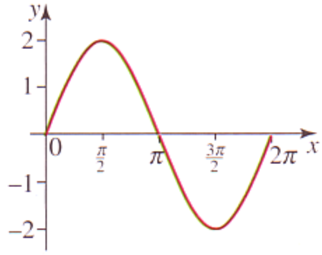
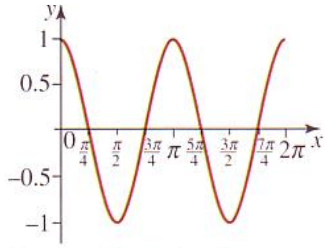
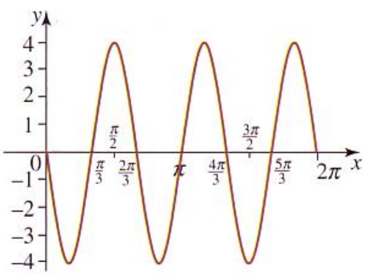
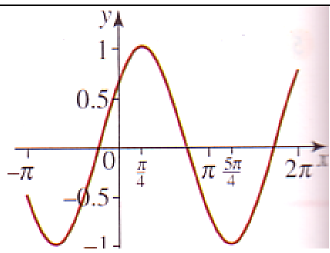
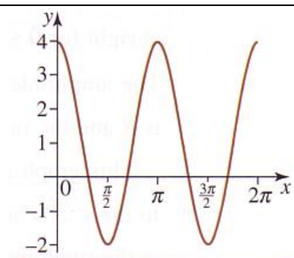
Range: $[-a, a]$
Domain: R

Amplitude $= |a|$

Period $= \dfrac{2\pi}{n}$

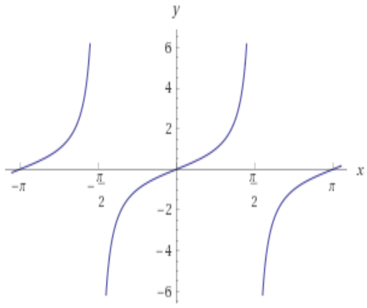

$f(x) = \sin x, -2\pi \le x \le 2\pi$


$f(x) = \cos x, -2\pi \le x \le 2\pi$

Transformation	Formula	Graph
Dilation in y-direction	$y = a\sin x$	 $f(x) = 2\sin x, 0 \le x \le 2\pi$
Dilation by $\frac{1}{n}$ in the x-direction	$y = \sin nx$	 $f(x) = \cos 2x, 0 \le x \le 2\pi$
Reflection in the y-axis	$y = \sin -x$	
Horizontal translation	$y = \sin(x-b)$	
Vertical translation	$y = \sin x + c$	 $f(x) = 3\cos 2x + 1, 0 \le x \le 2\pi$

Tangent Function

The basic formula is $y = a\tan n(x-b)+c, y = a\cos n(x-b)+c$

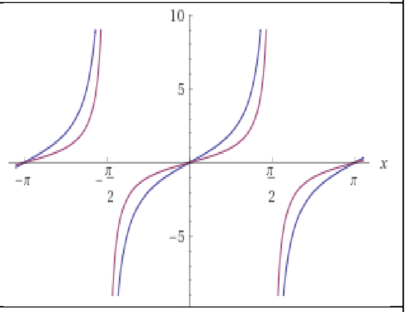
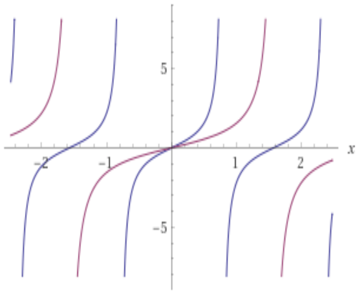
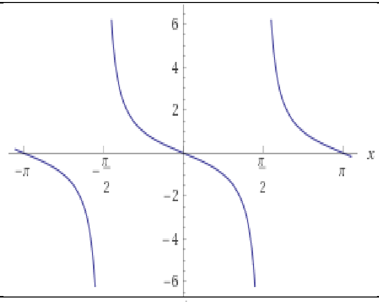
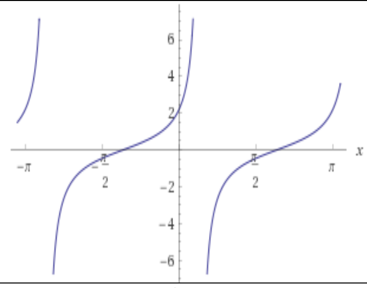


Range: R

Domain: $R\backslash\{x: x = \dfrac{(2k+1)\pi}{2n}\}$

Period $= \dfrac{\pi}{n}$

Asymptotes at $x = \left(\dfrac{(2k+1)\pi}{2n}\right) - b$, where $k = \{0, \pm 1, \pm 2 \ldots\}$

Transformation	Formula	Graph
Dilation in y-direction	$y = a\tan x$ 2 tan(x) tan(x)	
Dilation by $\frac{1}{n}$ in the x-direction	$y = \tan nx$ 2 tan(x) tan(x)	
Reflection in the y-axis	$y = \tan -x$ $y = -\tan x$	
Horizontal translation	$y = \tan(x-b)$	
Vertical translation	$y = \tan x + c$