Elementary Algebra

Basic Algebra

Laws of Divisibility

1. A number is divisible by 3 if the sum of all the individual digits is evenly divisible by 3.
2. A number is divisible by 4 if the number formed by the last two digits is divisible by 4.
3. A number is divisible by 6 if it is evenly divisible by both 2 and 3.
4. To determine if a x is divisible by 7, take the last digit off x, then double that digit and subtract the doubled number from (x minus final digit). If the result is evenly divisible by 7 (e.g. 14, 7, 0, -7, etc.), then the number is divisible by seven. This may need to be repeated several times.
5. A number is divisible by 8 if the number formed by the last three digits is divisible by 8.
6. A number is divisible by 9 if the sum of all the individual digits is evenly divisible by 9.

Fractions

Addition

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$$

Addition with different bases

$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$$

Example 1

Simplify the following expression:

$$\begin{aligned}\frac{x}{y}-\frac{x}{x+y}&=\frac{x(x+y)}{y(x+y)}-\frac{xy}{y(x+y)}\\&=\frac{x(x+y)-xy}{y(x+y)}\\&=\frac{x^2+xy-xy}{yx+y^2}\\&=\frac{x^2}{y(x+y)}\end{aligned}$$

Example 2

Simplify the following expression:

$$\begin{aligned}\frac{4}{a}-\frac{2}{a(a+2)}&=\frac{4a(a+2)}{a^2(a+2)}-\frac{2a}{a^2(a+2)}\\&=\frac{4a(a+2)-2a}{a^2(a+2)}\\&=\frac{4a^2+8a-2a}{a^2(a+2)}\end{aligned}$$

$$= \frac{4a^2 + 6a}{a^2(a+2)}$$
$$= \frac{2a(2a+3)}{a^2(a+2)}$$
$$= \frac{4a+6}{a(a+2)}$$

Multiplication

$$\frac{a}{c} \times \frac{c}{d} = \frac{ac}{cd}$$

Example 3

Simplify the following expression:

$$\left(\frac{1}{x} - \frac{1}{y}\right) \times \frac{1}{x-y} = \left(\frac{y}{xy} - \frac{x}{xy}\right) \times \frac{1}{x-y}$$
$$= \frac{y-x}{xy} \times \frac{1}{x-y}$$
$$= \frac{y-x}{xy(x-y)}$$
$$= \frac{-(x-y)}{xy(x-y)}$$
$$= -\frac{1}{xy}$$

Example 4

Simplify the following expression:

$$\frac{x+y}{x^2+4y^2} \times \frac{6y-3x}{2x+2y} = \frac{(x+y)(6y-3x)}{(x^2+4y^2)(2x+2y)}$$
$$= \frac{3(x+y)(2y-x)}{2(x+2y)(x-2y)(x+y)}$$
$$= \frac{-3(x-2y)}{2(x+2y)(x-2y)}$$
$$= -\frac{3}{2(x-2y)}$$

Division

$$\frac{a}{c} \div \frac{c}{d} = \frac{a}{c} \times \frac{d}{c}$$

Example 5

Simplify the following expression:

$$\frac{6a}{a-5} \div \frac{a+5}{a^2-25} = \frac{6a}{a-5} \times \frac{a^2-25}{a+5}$$
$$= \frac{6a(a^2-25)}{(a-5)(a+5)}$$
$$= \frac{6a(a^2-5^2)}{(a-5)(a+5)}$$

$$= \frac{6a(a^2 - 5^2)}{(a^2 - 5^2)}$$
$$= 6a$$

Example 6

Simplify the following expression:

$$\frac{x^2 - y^2}{\frac{1}{x} + \frac{1}{y}} = \frac{x^2 - y^2}{\frac{y}{xy} + \frac{x}{xy}}$$
$$= \frac{x^2 - y^2}{\frac{y + x}{xy}}$$
$$= \frac{x^2 - y^2}{1} \times \frac{xy}{y + x}$$
$$= \frac{xy(x^2 - y^2)}{y + x}$$
$$= \frac{xy(x - y)(x + y)}{x + y}$$
$$= xy(x - y)$$

Fractions in Fractions

$$\frac{\frac{a}{b}X}{\frac{c}{d}Y} = \frac{adX}{cbY}$$

Simplifying Fractions

To simplify a fraction, divide it by the Greatest Common Factor of the numerator and the denominator. To find the GCF of b and d, use the Euclidean Algorithm:

$$\frac{b}{d} = x_1\ R(r_1)$$
$$\frac{d}{r_1} = x_2 R(r_2)$$
$$\frac{r_1}{r_2} = x_3 R(r_3)$$

Repeat until $r_i = 0$

then $r_{i-1} = \text{GCF}$

Example 7

Find the Greatest Common Factor of 108 and 128

$$let\ 128 = b, 108 = d$$
$$\frac{128}{108} = 1\ R(20)$$
$$\frac{108}{20} = 5\ R(8)$$
$$\frac{20}{8} = 2\ R(4)$$
$$\frac{8}{4} = 2\ R(0)$$

$$\therefore GCF = 4$$

Adding Mixed Numbers

$$a\frac{b}{c} + d\frac{e}{c} = a + d + \frac{b+e}{c}$$

When adding or subtracting mixed numbers, do the whole number and fraction parts separately, then convert the improper fraction to a mixed number and add it to the whole number.

Example 8

$$\begin{aligned} 5\frac{11}{15} + 3\frac{7}{9} &= 8 + \frac{99}{135} + \frac{105}{135} \\ &= 8 + \frac{204}{135} \\ &= 8 + 1\left(\frac{69}{135}\right) \\ &= 9 + \frac{69}{135} \\ &= 9\frac{23}{45} \end{aligned}$$

Dividing Mixed Numbers

$$\frac{a}{c} \div x = \frac{a}{c} \div \frac{x}{1} = \frac{a}{c} \times \frac{1}{x} = \frac{a}{xc}$$

When multiplying or dividing by whole numbers, first convert them to fractions.

Surds

Multiplication

$$\sqrt{x} \times \sqrt{y} = \sqrt{xy}$$

Division

$$\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{x/y}$$

Addition and Subtraction

$$a\sqrt{x} \pm b\sqrt{x} = (a \pm b)\sqrt{x}$$

Rationalising

$$\begin{aligned} \frac{a}{b+\sqrt{x}} &= \frac{a}{b+\sqrt{x}} \times \frac{b-\sqrt{x}}{b-\sqrt{x}} \\ &= \frac{a\left(b+\sqrt{x}\right)}{b^2+x} \end{aligned}$$

Polynomials

A polynomial is an expression that consists of terms with only positive, whole powers of x.

Thus, the following are not polynomials:

$x^{\frac{9}{2}} + x^3 - 6$
$7 - 3xy + 4x^2$
$xy + y^4 + 12$
$3x^3 + 6 + \sqrt{x}$

Expanding Polynomials

Expanding Different Terms

$(x + a)(y + b) = x(y + b) + a(y + b) = xy + xb + ay + ab$

Expanding Like Terms

$(x + a)(x - a) = x^2 - a^2$

Expanding Squared Terms

$(x + a)^2 = x^2 + 2ax + a^2$
$(x - a)^2 = x^2 - 2ax + a^2$

Expanding Cubed Terms

$(x + a)(x^2 - 2ax + a^2) = x^3 + a^3$
$(x - a)(x^2 + 2ax + a^2) = x^3 - a^3$

Example 1

Simplify the following expression:

$$\begin{aligned}(x - y)(x + y - 2) &= x^2 + xy - 2x - xy - y^2 + 2y \\ &= x^2 - 2x + 2y - y^2\end{aligned}$$

Example 2

Simplify the following expression:

$$\begin{aligned}(x + 1)(x - 2)(x + 3) &= (x + 1)(x^2 - 2x + 3x - 6) \\ &= (x + 1)(x^2 + x - 6) \\ &= (x^3 + x^2 - 6x + x^2 + x - 6) \\ &= x^3 + 2x^2 - 5x - 6\end{aligned}$$

Example 3

Simplify the following expression:

$$\begin{aligned}a^4 + b^4 &= (a^2)^2 + (b^2)^2 \\ &= (a^2 - b^2)(a^2 + b^2) \\ &= (a - b)(a + b)(a^2 + b^2)\end{aligned}$$

The Binomial Theorem

The binomial theorem is used to expand expressions of the form $(ax + b)^n$

The formula is as follows:

$$(ax+b)^n = \binom{n}{0}(ax)^n b^0 + \binom{n}{1}(ax)^{n-1}b + \cdots + \binom{n}{n-1}(ax)b^{n-1} + \binom{n}{n}(ax)^0 b^n$$

Note that:

$$\binom{n}{r} = nC_r = \frac{n!}{(n-r)!\,r!}$$

$= \text{number of ways of choosing } r \text{ objects from a group of } n \text{ (order important)}$

Coefficients can be found using Pascal's Triangle

$(x+a)^0$	$\binom{0}{0}$	1
$(x+a)^1$	$\binom{1}{0}\ \binom{1}{1}$	1 1
$(x+a)^2$	$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$	1 2 1
$(x+a)^3$	$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{0}$	1 3 3 1
$(x+a)^4$	$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$	1 4 6 4 1

Thus, the $(r+1)$th term will be $\binom{n}{r}(ax)^{n-r}b^r$

Example 1

Expand $(2x-3)^4$using the binomial theorem

$$(2x-3)^4 = \binom{4}{0}(2x)^4(-3)^0 + \binom{4}{1}(2x)^3(-3)^1 + \binom{4}{2}(2x)^2(-3)^2 + \binom{4}{3}(2x)^1(-3)^3$$
$$+ \binom{4}{4}(2x)^0(-3)^4$$
$$= (1)(16x^4)(1) + (4)(8x^3)(-3) + (6)(4x^2)(9) + (4)(2x)(-27) + (1)(1)(81)$$
$$= 16x^4 - 19x^3 + 216x^2 - 216x + 81$$

Example 2

Expand $(x-2y)^4$using the binomial theorem

$$(x-2y)^4 = \binom{4}{0}(x)^4(-2y)^0 + \binom{4}{1}(x)^3(-2y)^1 + \binom{4}{2}(x)^2(-2y)^2 + \binom{4}{3}(x)^1(-2y)^3$$
$$+ \binom{4}{4}(x)^0(-2y)^4$$
$$= (1)(x^4)(1) + (4)(x^3)(-2y) + (6)(x^2)(4y^2) + (4)(x)(-8y^3) + (1)(1)(16y^4)$$
$$= x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4$$

Example 3

Expand $\left(\frac{2}{x^2} + x\right)^4$ using the binomial theorem

$$\left(\frac{2}{x^2}+x\right)^4 = \binom{4}{0}\left(\frac{2}{x^2}\right)^4 (x)^0 + \binom{4}{1}\left(\frac{2}{x^2}\right)^3 (x)^1 + \binom{4}{2}\left(\frac{2}{x^2}\right)^2 (x)^2 + \binom{4}{3}\left(\frac{2}{x^2}\right)^1 (x)^3$$
$$+ \binom{4}{4}\left(\frac{2}{x^2}\right)^0 (x)^4$$
$$= (1)\left(\frac{16}{x^8}\right)(1) + (4)\left(\frac{8}{x^6}\right)(x) + (6)\left(\frac{4}{x^4}\right)(x^2) + (4)\left(\frac{2}{x^2}\right)(x^3) + (1)(1)(x^4)$$
$$= \frac{16}{x^8} + \frac{32}{x^5} + \frac{24}{x^2} + 8x + x^4$$

Completing the Square

Completing the square is a technique for converting a quadratic polynomial of the form $ax^2 + bx + c$ into the form $a(x-h)^2 + k$. This can be very useful when graphing the function or finding the turning point.

The algorithm for this is simply to add and subtract $\left(\frac{b}{2}\right)^2$, then simplify and convert the result to perfect square form.

Example

Solve $2x^2 - 4x - 5 = 0$ by expressing it in the form $y = a(x-h)^2 + k$

$$2x^2 - 4x - 5 = 2\left(x^2 - 2x - \frac{5}{2}\right)$$
$$2\left(x^2 - 2x - \frac{5}{2}\right) = 2\left(x^2 - 2x - \frac{5}{2}\right) - 1 + 1$$
$$0 = 2(x^2 - 2x + 1) - \frac{5}{2} - 1$$
$$= 2([x-1][x-1]) - \frac{7}{2}$$
$$= 2(x-1)^2 - 7$$

Factorising Polynomials

The Factor Theorem

This is a quick and easy way to see if one polynomial is a factor of another.

If $P(a) = 0$, then $(x-a)$ is a factor of $P(x)$

If $P\left(-\frac{b}{a}\right) = 0$, then $(ax+b)$ is a factor of $P(x)$

Example

Determine whether $(x-3)$ is a factor of $P(x) = 2x^3 - 4x^2 - 3x - 8$

$$P(3) = 2(3)^3 - 4(3)^2 - 3(3) - 8$$
$$= 2(27) - 4(9) - 9 - 8$$
$$= 54 - 36 - 17$$
$$= 1$$
$$\neq 0$$
$$\therefore (x-3) \text{ is not a factor}$$

The Quadratic Formula

The quadratic formula is used to factorise quadratic polynomials, and hence find their x-intercepts.

$$y = ax^2 + bx + c$$
$$= a(x - x_1)(x - x_2)$$
$$\text{where } x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant is the term in the square root, namely $b^2 - 4ac$

- If $b^2 - 4ac > 0$, then there will be two x-intercepts
- If $b^2 - 4ac > 0$ and is also a perfect square, then there will be two rational x-intercepts
- If $b^2 - 4ac = 0$, then there will be one x-intercept at the turning point
- If $b^2 - 4ac < 0$, then there will be no x-intercepts

Example

Factorise the equation $y = 2x^2 + 3x - 10$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-10)}}{2(2)}$$
$$x = \frac{-3 \pm \sqrt{9 - -80}}{4}$$
$$x = \frac{-3 \pm \sqrt{89}}{4}$$
$$x = \frac{-3 - \sqrt{89}}{4}, x = \frac{-3 + \sqrt{89}}{4}$$
$$x = \frac{-3 - 9.434}{4}, x = \frac{-3 + 9.434}{4}$$
$$x = -3.11, x = 1.61$$

Long Division

To divide one polynomial by another, the following method is used:

$$P(x) = Q(x) + \frac{R(x)}{D(x)}$$

where: $Q(x)$ is the quotient
$R(x)$ is the remainder
$D(x)$ is the divisor

If $P(x)$ is divided by$(ax + b)$, then the remainder is $P\left(-\frac{b}{a}\right)$

Example

Divide the expression $x^3 - 12x^2 - 42$ by $x - 3$

1. Divide the first term of the numerator by the highest term of the denominator. Place the result above the bar ($x^3 \div x = x^2$).

$$\begin{array}{r|l} & x^2 \\ \hline x - 3 & x^3 - 12x^2 + 0x - 42 \end{array}$$

2. Multiply the denominator by the result you just obtained (the first term of the eventual quotient). Write the result under the first two terms of the numerator ($x^2 \cdot (x - 3) = x^3 - 3x^2$).

$$\begin{array}{r|l} & x^2 \\ \hline x-3 & x^3 - 12x^2 + 0x - 42 \\ & x^3 - 3x^2 \end{array}$$

3. Subtract the product you just obtained from the appropriate terms of the original numerator, and write the result underneath. This can be tricky at times, because of the sign. ($(x^3 - 12x^2) - (x^3 - 3x^2) = -12x^2 + 3x^2 = -9x^2$) Then, "bring down" the next term from the numerator.

$$\begin{array}{r|l} & x^2 \\ \hline x-3 & x^3 - 12x^2 + 0x - 42 \\ & \underline{x^3 - 3x^2} \\ & \qquad -9x^2 + 0x \end{array}$$

4. Repeat the previous three steps, except this time use the two terms that you have just written as the numerator.

$$\begin{array}{r|l} & x^2 - 9x \\ \hline x-3 & x^3 - 12x^2 + 0x - 42 \\ & \underline{x^3 - \ \ 3x^2} \\ & \qquad -9x^2 + 0x \\ & \qquad \underline{-9x^2 + 27x} \\ & \qquad\qquad -27x - 42 \end{array}$$

5. Repeat step 4. This time, there is nothing to "pull down".

$$\begin{array}{r|l} & x^2 \ - 9x \quad - 27 \\ \hline x-3 & x^3 - 12x^2 + 0x - 42 \\ & \underline{x^3 - \ \ 3x^2} \\ & \qquad -9x^2 + 0x \\ & \qquad \underline{-9x^2 + 27x} \\ & \qquad\qquad -27x - 42 \\ & \qquad\qquad \underline{-27x + 81} \\ & \qquad\qquad\qquad -123 \end{array}$$

6. Therefore the answer is:

$$\frac{x^3 - 12x^2 - 42}{x - 3} = x^2 - 9x - 27 - \frac{123}{x - 3}$$